***Round Table***
***Shaping the 21st Century***
***3ECM, 11-14 July 2000, Barcelona***

---

**Contribution of** *Jean-Pierre Bourguignon, Paris, France*

The comments I would like to add to the very focused presentations of my fellow speakers are a bit more diffuse. Firstly, I feel a few words about fashions and trends in mathematics are in order. An excuse for this: from what I understand, this was one of the main topics Mikhael Gromov wanted to touch upon, had he participated in this round table. Unfortunately for you, I am likely to be less audacious than him on this theme. Diversity is another topic, which has already been mentioned before, I want to bring to your attention: how to deal with it? how to prepare our students for it? The last topic, for me a key for the future, is how to involve research mathematicians in the training of teachers, initial and continued. In my opinion, the mathematical community as a whole has not devoted enough attention to this matter.

Mathematicians tend to believe their activity is immune of fashions. As director of an Institute visited by both mathematicians and physicists, I can indeed testify that, as far as fashions are concerned, mathematicians have an attitude which differs from that of theoretical physicists.

Let me start with a historical example which, I think, deserves to be known. Some ten years ago, the Societé Mathématique de France commissioned Hélène Gispert, a historian of mathematics, to prepare a volume describing its early history. In an appendix to this volume she also presents a nice piece of history, namely a compilation of reports on all theses defended in France during the period

1870-1910. One can easily recognize that the fashion of the time was the theory of functions of one complex variable, a very noble subject indeed but, while reading the document, one soon realizes how overdone are the emphasis and the enthusiasm expressed in reports on works in this area. They should make one believe that all specialists of this domain made tremendous contributions to mathematics when, if some did, most others have completely disappeared from our sight. You will be very surprised to discover the very critical tone of the report on the thesis by Henri Poincaré, who worked on a more daring, but more long-living, subject, in sharp contrast with the others.

In fact it is extremely difficult for a discipline to be really insensitive to fashions. It may altogether be undesirable, since fashions are a way of making people open to indispensable changes. We have no choice but taking them into account, while carefully avoiding them creating inappropriate unbalance.

My feeling is that so far mathematics has kept its unity in a form which is rather adequate, and prepares it well for the future, because presently unity develops itself by means of a very dynamic process. Indeed, even if today one can identify, and therefore name, some subdisciplines of mathematics, probably some fifty to seventy of them, the main point about them is that the connections they enjoy with one another are in constant evolution, paving the way to interesting cross-fertilizations. This permanent redefinition of the topology of mathematics is for me one of the very important features that now shapes the mathematical landscape, and makes the discipline healthy. As you can imagine, this makes it absolutely necessary that we carefully explain this process to our young students, and make the necessary adjustments in the courses we offer.

Let me try and dare to mention what I see as deep trends in the mathematical sciences. Identifying them is an extremely difficult endeavour, and maybe a risky one, since, when doing this, one is almost sure of making wrong guesses. Nevertheless, the discussions making such a statement can provoke may still have a fruitful impact. In fact I would like to give two examples borrowed from the past, and then try and draw some lessons from them. Here are the two examples.

**Taking the paradigma of quantum mechanics seriously**
The first example deals with the behaviour of mathematicians with respect to quantum mechanics. As you know, the idea of quanta was introduced by physicists in order to deal with fundamental natural phenomena that were relevant to their discipline. Some of the names connected to this story were quoted by Alexandra Bellow a moment ago. In a sense, mathematicians got quickly content with the fact that the whole framework of equations of quantum mechanics relies on sophisticated mathematics. In this context wave functions becomeelements in a vector space $L^2$, Hilbert spaces are very useful, operator theory very important, as are C*-algebras, developed by John von Neumann, and so on and so forth. In a sense, for a long time, the way mathematicians perceived their relation to quantum mechanics was limited to bringing a technical apparatus to physicists who dealt with the theory. But mathematicians did not really take seriously into account the building blocks of quantum mechanics which, before being a

predictive theory, has required a deep rethinking of all physical concepts. Note that this very thorough refoundation is still posing problems (because, as you know, one cannot just quantize once, but must do it twice, a process which requires going up to Field Theory and the sophistication that goes with it). Indeed quantum mechanics does change completely a number of the very basic paradigms of physics. It took mathematicians a very long time to take such an idea seriously, i.e. to engage in a parallel rethinking of mathematics in this light.

Of course, it is not a quantum mathematics that mathematicians are developing, but something different, namely transposing to mathematics what are some of the very basic new ideas that found quantum mechanics. This process started to take place at a large scale in the late eighties and early nineties mainly among mathematicians who were enjoying the closest contacts with physicists, in particular our Russian colleagues. They definitely understood all advantages that mathematicians could draw from this exercise, in particular the need to consider seriously non-commutativity in many areas of mathematics where this was not yet the case.

I was very pleased to see that in this Congress, and especially today, many lectures incorporated some ideas that had been borrowed from other disciplines, in particular from quantum mechanics. This showed they can be absorbed by mathematicians, and give rise to new mathematical developments or pose challenging problems.

**Rethinking the basis of linear algebra through spinors**
The other example mixes discussing fashion and trend. Maybe you will be surprised if I tell you that the notion of spinors really goes back to Henry Earle Clifford, while

Élie Cartan was the person who formally defined spinors for the first time. He did this while doing a systematic classification of representations of the orthogonal groups. To his surprise, he noticed that besides the ones he expected to find, there were some exotic ones. This fact remained unexploited in a dark corner, until Paul Adrien Maurice Dirac, while working at developing a relativistictheory of the electron, needed objects which finally turned out to be spinors. All of a sudden wave functions of electrons could not just be functions, but had to be spinor fields.

Since then, only seldomly did mathematicians take spinors seriously, while, in physics, they were confirmed as absolutely fundamental objects modelling basic constituents of matter. Their first real occurence in mathematics was in the web of the Atiyah-Singer index theorem, because there many things could be brought back to the Dirac operator, which all of a sudden started to become a central object in mathematics too. Even then the need to introduce and work with spinors was not overwhelming. In some sense, in order to see this change we had to wait the wave created by the fashion of Seiberg-Witten theory, which caught the attention of many mathematicians, mostly because, using it away from the purely physical initial motivation of the theory, a lot of very hard computations in topology could be muchsimplified. A number of important applications to the understanding of the geometry of 4-dimensional manifolds could be drawn from it, and all of a sudden mathematicians could not escape dealing with spinors, and gaining some intimacy with them. The fashion is already fading away, and they have not yet achieved in mathematics a role analogous to the one won in physics.

My point here is that mathematicians may have not yet gone far enough and put enough thoughts into refounding geometry using spinors, as physicists did following the advice of Roger Penrose, who actually would like them to go even further. Maybe the time has come for some fundamental objects of mathematics to be rethought in this light too. The emergence of ``super''-objects in mathematics can certainly be seen as a first step in this direction but mathematicians have not yet seen how they could fully take advantage of supersymmetries for example.

**Possible Trends**

To introduce the discussion of tendencies in mathematics, I am tempted to take up what Professor Neunzert said and claim that mathematics may presently be undergoing a phase transition.

My view here is that mathematicians will have more and more to consider the paradigms of other disciplines and see how they can shed a new light on some of the mathematical paradigms. Quite often this means we will have to rethink large portions of mathematics using other lines of thought. The discipline to which I have personally been looking at for some time is Biology, for reasons again connected to my own duties as director of an Institute. Although the IHÉS has not yet hired a biologist as permanent professor, I am very pleased to see that one of them, Mikhael Gromov to name him, has decided to devote his full energy to the study of biological problems from the point of view of a mathematician. (Along these lines, this Fall, the IHÉS starts a group working in functional genomics under his leadership, and of course with the cooperation and involvement of a group of biologists.)

For me an important remark to be made about Biology is that its basic concepts are extremely manifold. As a result, it is difficult to speak about Biology as a single science. It has been pointed out earlier that the future of Biology may be to become an organizing centre for many other natural sciences, and also humanities. Mathematicians will have to find their place in this new landscape. This challenge of a special nature will probably force them to forge new definitions, develop new tools, and likely also adopt new attitudes. In particular it is foreseeable that statistics will have to be dressed with new clothes and get more closely connected to many other areas of mathematics than it is now. The quest for a deeper understanding of biological objects will play an important role in this process that can eventually change the position of statistics within the mathematical sciences.

This morning, Professor Hans Föllmer explained us how much
mathematicians could benefit from getting involved with some of
the problems coming from the Finance industry, as he said. But the
same could be said of Economics, although it is neither the same
problems nor the same paradigms. This shows that mathematicians
must consider that some roots of their discipline, although it
has sound underpinnings, still remain to be explored. Hopefully, this will open the way to defining, and

introducing among mathematicians, other ways of thinking about mathematics.

These are the general tendencies I wanted to offer to your reflection. All this at a time where mathematicians still go on developing techniques and proving new theorems for themselves very successfully, and hopefully so. In some cases, on this new mathematical edifices new technologies will be built, as Professor Yves Meyer showed us very convincingly this morning. Which will be relevant is hard to predict, and will remain so.

To summarize, along the paths mathematicians ought to pursue, they will need first to understand the paradigms of other sciences at a deeper level, and then to see how mathematicians can make use of them for the benefit of their own discipline. This will almost certainly not happen by just copying what others have been doing but will require a rethinking of some parts of mathematics. This brings me to my second point about

**Possible Impact on the Training of Students**

For me, a lot of the future of mathematics will be rooted in this diversity. When training students, we all know that at some point we have to make them become specialists of something, in order that they can finally make a living of their mathematical knowledge, either in academia or in the world of companies. How can we achieve the proper balance in the training of young people between making them experts in a field and having a broad culture in mathematics? How can we nurture these diversified profiles, namely being able to understand a large number of facets of the discipline and, as I suggested earlier, also having a reasonable knowledge of other disciplines? There is no way one can model, or discuss the pertinence of models for, other sciences without having a basic acquaintance with them. Certainly, exposing a large enough number of students to the ways of thinking of other disciplines will become crucial. Part of the challenge, by no means the least trivial, is to keep their curiosity while they acquire some basic knowledge of other parts of mathematics and science in general. How can we educate ourselves further, after we have been doing something specialised, and successfully begin to study some new domains? What kinds of documents have we at our disposal for that purpose? What are the forums in which we can learn? How can we meet other people in a professionally efficient way? Are there enough of these interdisciplinary meetings, where people can really exchange in depth about their own disciplines?

At this moment I feel the community has not adequately adressed these problems. There are indeed few places where it is known one can meet people from other disciplines in a context where the exchange is likely to be fruitful. I am pleased that many visitors of the IHÉS tell me the Institute has preserved this forum aspect. To keep this happening is not easy because, in view of the differences in sociology that I briefly mentioned earlier, compromises must be made. For me it is very important that we, as a community, think on how we can foster these interactions. Probably part of this activity will be done on a purely personal basis, that is by getting access to the appropriate web sites, to the proper books, to videos or simply by talking face to face to the right people. Part of the solution will also come from setting up appropriate conferences or forums in which this type of exchange may take place more easily.

**Bridging the gap with teachers**

I am now coming to my last point which is the need for efficient connections between research mathematicians and teachers. It is undoubtely a key issue if one is to seriously consider the moto of this Congress, namely "Shaping the XXIst century". This question has already been addressed by Prof. Neunzert. For me, being able to attract very good minds in the coming years to study mathematics is a real challenge. The task will not be easy. If one considers the question Professor Neunzert has been asking us, namely, "whether a young child who is very ambitious about contributing to the development of society will choose to study mathematics",one is forced to recognize the answer is not obvious. The only way we may have a chance of succeeding to attract these students is by having excellent ambassadors of our discipline. Surely, the best way for that is to develop sound and confident relations with schoolteachers. This is certainly possible, but requires real commitments on our part.

I see two major difficulties for achieving this. First of all, although most of the time teachers have kept the basic love of mathematics and interest in the discipline that made them chose this profession, in too many countries we as research mathematicians give them the feeling that we are looking down upon them. Indeed they too often have the impression that we view them as people who have not been able to achieve professionally what we have achieved ourselves. We are completely wrong in not worrying about this, because in fact being a teacher in a secondary school, or even in a primary school, is a job in itself, and a very demanding one. How many of us have been turned on by one of them, therefore giving a decisive impetus to our love for mathematics? To the contrary we need to give them more opportunities of getting the feeling that they are successful.

A second important point: we have to make them aware of the fact that mathematics is alive. It is true that the development of mathematics goes beyond the level at which they stopped their own studies. One must create opportunities for teachers to meet professional mathematicians. Again how can we do that? How can we write and produce documents of all kinds to ease this contact? Of course the development of internet and websites and so on creates fantastic opportunities, and it is now quite simple to make documents widely available. Who will take initiatives in this direction? It looks indispensable to help personal initiatives come to fruition. Again I fear that, as a community, we are not dealing properly with this issue.

In many countries, and France is one of them, the debate about contacts with secondary school teachers was for a long time limited to whether research mathematicians should make compromises with didacticians. As a result, the main discussion was focused on the content of programmes and pedagogy. For me, if the problem of the transmission of knowledge in schools has of course a purely technical part, since one must make sure students achieve a basic technical mastery, another dimension of the learning process is about transmitting the enthusiasm for and the curiosity about the discipline. This necessarily requires the eagerness of teachers to explore new avenues with students. For this we definitely have to provide tools, and show our willingness, as research mathematicians, to participate in this venture.

Unfortunately, I am not sure we have enough realized how important it is to devote more energy to deal with this issue for the future of our discipline. There is little time to deal with the formidable challenges that lie ahead of us in view of the massive turn over of mathematicians coming up in the first decade of the century. We cannot let this fantastic opportunity pass. We must confront the issues of the broadening of our discipline in partnership with teachers at all levels.