Problem 1
Hint Consider a rectangle with an odd number of sides, such as:
```
x-----x
|     |
x--x--x
```
Solution
```
A-----B-----C
|     |     |
D--E--F--G--H
|  |     |  |
I--J-----K--L
```
If a curve intersects the boundary of a region R (such as ABFED), then it moves from inside R to outside or vice versa. Hence if R has an odd number of edges (like ABFED) then a curve intersecting all of them just once must have one endpoint inside R. But there are four such regions (ABFED, BCHGF, EFGKJ and the outside of ABCHLKJID) and only two endpoints. Note that we can easily intersect all edges but one. For example, start above AB, then cross successively AB, AD, DI, DE, EF, EJ, IJ, JK, GK, KL, HL, GH, CH, BC, FG.

Problem 2
Hint The center of the required circle is the center of the rectangle.
Solution Let O be the center of the rectangle. Let r = (a+c)/2 = (b+d)/2. The required circle has center O, radius r. Let an outer common tangent touch the circle center A at W, and the circle center C at X. Let P be the midpoint of WX, then OP is parallel to AW and CX and has length r, hence the circle center O touches AW at P. Similarly for the other common tangents.

Problem 3
Hint Consider the sequence starting with 999981.
Solution Let n be the smallest number in the sequence and m the smallest with last digit 0. m and m+10 have different digit sums unless (possibly) the penultimate digit of m is 9, but in that case m+10 and m+20 have different digit sums. So two of m, m+10, m+20 are sure to have different digit sums. Hence at least one has a digit sum not congruent to 1 mod 11. Adding the appropriate final digit gives a number whose digit sum is divisible by 11. This number lies in the range m to m+29 and m<=n+9. Hence the result. n=999981 shows it is best possible.

Problem 4
Hint Pigeon hole principle.
Solution
(a)
```
**..
*.*.
.**.
...*
```
Pick any two rows. The unpicked stars lie in different columns.
(b) If there is a row with at least 3 stars, pick it. That leaves at most 3 stars, pick the row for one and the columns for the others. Now assume no row has more than 2 stars. 6 stars in <6 rows, so we can pick a row with 2 stars. That leaves 4 stars in 3 rows, so we can pick another row with 2 stars. That leaves 2 stars. Pick their columns. [This glosses over the case of <6 stars. In this case we can add extra stars to make the number up to 6. Now the procedure above deals with the original stars and the extra stars, and in particular with the original stars.]

Problem 5

Hint
(a) Trivial if abcd not 1. In the non-trivial case, show the largest diverges.
(b) n iterations suffice.
Solution
(a) Let Q_0 be the original quadruple (a, b, c, d) and Q_n the quadruple after n transformations. If abcd>1, then the products form a strictly increasing sequence, so return is impossible. Similarly if abcd<1. So we must have abcd=1.
Let the largest of the four values of a quadruple Q be M(Q). If a member of Q_1 is not 1, then $M(Q_1)>1$. Q_3 consists of the elements of Q_1 squared and permuted, so $M(Q_3) = M(Q_1)^2$. Hence the sequence $M(Q_1)$, $M(Q_3)$, $M(Q_5)$, ... increases without limit. This means no return is possible, because a return would lead to the values cycling.

(b) After r<n transformations, the first number of the n-tuple is the product $a_1^{(r|0)}a_2^{(r|1)} \ldots a_{r+1}^{(r|r)}$, where (r|i) denotes the binomial coefficient. [This is an easy induction.] Hence after $n=2^k$ transformations it is a_1^2 times the product $a_2^{(n|1)} \ldots a_n^{(n|n-1)}$. So it is sufficient to prove that (n|i) is even for n a power of 2 and 0<i<n. But observe that (n|i) = (n-1|i) n/(n-i) and n is divisible by a higher power of 2 than n-i.

Problem 6
Hint
(a) The locus is usually a circle.
(b) Answer: 5. Given any P we have CP <= AP + BP.
Solution
(a) Represent A, B as complex numbers $z_1 + w_1e^{it}$, $z_2 + w_2e^{it}$. Then C is $(z_1 + w_1e^{it}) + (z_2 + w_2e^{it} - z_1 - w_1e^{it})\, e^{i\,pi/3}$, which is also of the form $z + w\, e^{it}$.

However, there is one subtlety. There are actually two circles possible for C depending on which side of AB we place it. The continuity requirement means that C is normally confined to one of the circles. However, if A and B ever coincide then C may be able to switch to the other circle.

If we regard "moves continuously" as allowing a discontinuous velocity, then a switch is always possible (provided A and B coincide).

(b) Answer: 5.

P must be the opposite side of AB to C (or we could increase CP, whilst keeping AP and BP the same, by reflecting in AB). Similarly it must be on the same side of AC as B, and on the same side of BC as A. For any P in this region the quadrilateral APBC is convex and hence satisfies Ptolemy's inequality CP.AB<=AP.BC + BP.AC, with equality iff APBC is cyclic. But AB = BC = CA, so we have CP<=AP + BP = 5 with equality iff P lies on the arc AB of the circle ABC. Note that there is just one such point, because the locus of P such that BP=1.5 AP is a circle which cuts the arc just once.

Ptolemy's inequality for 4 points A, B, C, D: AB.CD + BC.AD >= AC.BD with equality iff ABCD is a cyclic quadrilateral (meaning A, B, C, D lie on a circle in that order).

Proof
Take E inside ABCD such that angle DAE = angle CAB and angle ADE = angle ACB. Then ADE and ACB are similar, so DE/CB = AD/AC and hence BC.AD = AC.DE. It also follows

that AE/AB = AD/AC. But we also have angle EAB = angle DAC and hence AEB and ADC are also similar. So EB/AB = DC/AC, and hence AB.CD = AC.EB. Adding, we have: AB.CD + BC.AD = AC(BE + ED) >= AC.BD with equality iff E lies on BD, or equivalently ABCD is cyclic.

This glosses over one point. It only follows that angle EAB = angle DAC if ABCD is convex. For the convex case, we have that angles EAB = CAB + EAC and DAC = DAE + EAC, or EAB = CAB - EAC and DAC = DAE - EAC. Either way EAB = DAC. But in the non-convex case, we can have EAB = CAB + EAC and DAC = DAE - EAC (or - ... +) and hence the angles EAB and DAC are not necessarily equal.

Problem 7
Hint Consider the sum of all the numbers.
Solution The array has mn entries. Call an array that can be obtained by repeated changes a reachable array. A reachable array differs from the original only in that some or all of the signs of its mn entries may be different. There are at most 2 possibilities for each sign and hence at most 2^{mn} different reachable arrays. For each reachable array calculate the sum of all its entries. Take the reachable array with the largest such sum. It must have non-negative row and column sums, because if any such sum was negative, changing the sign of that row or column would give another reachable array with strictly greater total sum.

Problem 8
Hint There is a point with just one edge.
Solution Every point must have at least one edge. We show that there is a point with just one edge. Suppose the contrary, that every point has at least two edges. We now construct a path in which the same edge or point never appears twice. Starting from any point b, move along an edge to c. c is not already on the path, because otherwise the edge would join b to itself. Now suppose we have reached a point x not previously on the path. x has at least two edges, so it must have another one besides the one we used to reach it. Suppose this joins x to y. If y is already on the path, then we have two distinct ways of moving along edges from x to y: directly, or by backtracking along the path from x to y. But this is impossible, so y is not already on the path and we may extend the path to it. But this procedure allows us to construct a path containing more than the n distinct points available. Contradiction.
The result is now easy. Induction on n. Take a point with just one edge. Remove it and the edge. Then the remaining n-1 points satisfy the premise and hence have just n-2 edges.

Problem 9
Hint Start by finding integers e and f such that em+fn=0.
Solution Care is needed. Although easy, this is more awkward than it looks.
Let d=(m,n), the greatest common divisor of m and n. Let r=n/d, s=nhk - m/d, where h is any integer sufficiently large to ensure that s>0. Now rm+sn = mn/d + nnhk - mn/d = nnhk, which is a multiple of k. If e divides r, then it also divides rdhk = nhk. So if e divides r and s, then it also divides s - nhk = -m/d. But n/d and m/d are relatively prime, so e must be 1. Hence r and s are relatively prime.

Problem 10
Hint Answers: [N/2], [(N+1)/2], [N/2]. The tricky case is *R2*. If N is odd, A must leave a pile of 2.
Solution Answers: [N/2], [(N+1)/2], [N/2].
Suppose A leaves piles n, m with n<=m.

Under *R1*, B can certainly secure m by dividing the larger pile into 1 and m-1. He cannot do better, because if b is the biggest of the 4 piles, then the smallest is at most m-b. Hence A's best strategy is to leave [N/2], [(N+1)/2].
Under *R2*, if A leaves a=2, b=N-2, then B cannot do better than [N/2], because if he divides the larger pile into a,b with a<=b, then he takes a+1. A cannot do better, because if he leaves a,b with 3<=a<=b, then B can divide to leave 1, a-1, [b/2], [(b+1)/2]. Now if a-1>=[(b+1)/2], then B takes b>=[(N+1)/2]. If a-1<[(b+1)/2], then B takes a-1+[b/2]. But a-1>=2 and [b/2]>=[(b+1)/2]-1, so a-1+[b/2]>=1+[(b+1)/2], or B takes at least as many as A, so B takes at least [(N+1)/2].
Under *R3*, A's best strategy is to divide into [N/2],[(N+1)/2]. We have already shown that B can secure [(N+1)/2] and no more by following *R1* . He cannot do better under R2, for if he divides so that the biggest pile comes from [N/2], then the smallest does too and so he gets [(N+1)/2]. If he divides so that the biggest and smallest piles come from [(N+1)/2], then he gets only [N/2]. But one of these must apply, because if he divided so that the smaller from [N/2] was smaller than the smaller from [(N+1)/2], and the bigger from [N/2] was smaller than the bigger from [(N+1)/2], then [N/2] would be at least 2 less than [(N+1)/2] (which it is not).

Problem 11
Hint Any infinite sequence of natural numbers has a non-decreasing subsequence.
Solution Given any infinite sequence of natural numbers we can find a non-decreasing subsequence (proof below). So suppose the three sequences are a_i, b_i, and c_i. Take a non-decreasing subsequence of a_i. Suppose it is a_{i1}, a_{i2}, a_{i3}, ... Now consider the infinite sequence b_{i1}, b_{i2}, ... It must have a non-decreasing subsequence. Suppose it is b_{j1}, b_{j2}, ... Now consider the infinite sequence c_{j1}, c_{j2}, ... It must have a non-decreasing subsequence c_{k1}, c_{k2}, ... Each of the three sub-sequences a_{k1}, a_{k2}, ... , b_{k1}, b_{k2}, ... , c_{k1}, c_{k2}, ... is non-decreasing. So we may take, for example, m=k2 and n=k1.
[Proof that any infinite sequence of natural numbers has a non-decreasing subsequence: if the original sequence is unbounded, then we can take a strictly increasing subsequence. If not, then since there are only finitely many possible numbers not exceeding the bound, at least one of them must occur infinitely often.]

Problem 12
Hint Consider the center of a circle. Where must it lie if the circle intersects a square?
Solution If a circle with unit diameter intersects a unit square, then its center must lie inside an area 3+pi/4, namely an oval centered on the square and comprising: the original square, area 1; four 1 x 1/2 rectangles on the sides, total area 2; and four quarter circles at the corners, total area pi/4. So if it does not intersect any of the 120 unit squares, then it must avoid ovals with a total area of 120 x (3+pi/4) = 454.2. Of course, for many arrangements of the squares, these ovals might overlap substantially, but the worst case would be no overlap.
The circle is also required to lie inside the rectangle, so its center must lie outside a strip 1/2 wide around the edge, and hence inside an inner 19 x 24 rectangle, area 456. The total area of ovals is less, so they cannot cover it completely and it must be possible to place a circle as required.

Problem 13
Hint Compare the areas of A'B'A and DAB.
Solution Compare the triangles A'B'A and ADB. The base of A'B'A can be taken as A'A, which is the same length as AD. The height of A'B'A is AB' times *sin* B'AA', which is twice AB times *sin* BAD. So area A'B'A = 2 area ADB. Similarly, area B'C'B = 2 area BAC, area C'D'C = 2 area CBD, and area D'A'D = 2 area DCA. So adding, the area A'B'A + area C'D'C =

2 area ABCD, and area B'C'B + area D'A'D = 2 area ABCD. But ABCD = A'B'A + B'C'B + C'D'C + D'A'D + ABCD. Hence result.

Problem 14
Hint It is a straiht line - well almost!
Solution Let the common tangent meet C at S. Let X be the intersection of C and OP lying between O and P. PT = PO, hence angle POT = angle PTO, so angle OPT = 180 - 2 angle POT. But PT and OS are parallel, because both are perpendicular to the common tangent. Hence angle POS = 2 angle POT, so angle SOT = angle XOT. Hence TX is tangent to C, in other words T lies on the (fixed) tangent to C at X. Conversely, it is easy to see that any such point can be obtained (just take P such that PO = PT). Thus the required locus is the pair of tangents to C which are perpendicular to L.

Problem 15
Hint Express a_r in terms of a_1 and a_0.
Solution An easy induction gives $a_r = (2^r - 1)a_1 - (2^r - 2)a_0$ for r = 2, 3, ... , 100. Hence, in particular, $a_{100} = (2^{100} - 2)(a_1 - a_0) + a_1$. But a_1 and $(a_1 - a_0)$ are both at least 1. Hence result.

Problem 16
Hint 62-19
Solution If there were such values, then subtract the equation with x = 19 from the equation with x = 62 to get: $a(62^3 - 19^3) + b(62^2 - 19^2) + c(62 - 19) = 1$. But the left hand side is divisible by 62 -19 = 43, contradiction.

Problem 17
Hint Change an entry.
Solution If we change a -1 to 1, we affect the total number of rows and columns (containing an odd number of -1s) by 0, 2 or -2. After changing all the -1s we have total of 0. Hence the starting total must be even. So it cannot be n.

Problem 18
Hint The point at which the medians intersect is 2/3 of the way along each median..
Solution Let M be the midpoint of AB and X the midpoint of MB. Construct the circle center B, radius BC/2 and the circle diameter AX. If they do not intersect (so BC<AB/2 or BC>AB) then the construction is not possible. If they intersect at N, then take C so that N is the midpoint of BC. Let CM meet AN at O. Then AO/AN = AM/AX =2/3, so the triangles AOM and ANX are similar. Hence angle AOM = angle ANX = 90.

Problem 19
Hint How many numbers are there on the lhs of the inequality?
Solution Applying the arithmetic/geometric mean result to the 10 numbers gives the result immediately.

Problem 20
Hint Restrict M to a triangle formed by the center, a vertex and an adjacent midpoint of a side.
Solution Let X be the midpoint of AB and O the center of ABCDE. Suppose M lies inside AXO. Then $ME=r_3$. So we maximise r_3 by taking M at X, with distance 1.5590, and we minimise r_3 by taking M as the intersection of AO and EB with distance 0.8090. AXO is one of 10 congruent areas, so the required loci are (a) the 5 midpoints of the diagonals, and (b) the 5 midpoints of the sides.

Problem 21
Hint A number and its digit sum have the same remainder when divided by 9.
Solution x <= 9.1998 = 17982. Hence y <= the greater of 1+7+9+9+9=35 and 9+9+9+9=36. But 9 divides the original number and hence also x, y and z. Hence z=9.

Problem 22
Hint Consider the point X on BP with AX perpendicular to BP.
Solution Take X on AH so that BX is perpendicular to AH. Extend to meet HM at P'. Let N be the midpoint of AB. A, B, M and X are on the circle center N radius NA (because angles AMB and AXB are 90). Also MN is parallel to BC (because AMN, ACB are similar), so NM is perpendicular to MH, in other words HM is a tangent to the circle. hence P'M.P'M = P'X.P'B. Triangles P'XH and P'HB are similar (angles at P' same and both have a right angle), so P'H/P'X = P'B/P'H, so P'H.P'H = P'X.P'B. Hence P'H = P'M and P' coincides with P.

Problem 23
Hint Ignore the restrictions on CA.
Solution If we ignore the restrictions of CA, then the maximum area is 1, achieved when AB is perpendicular to BC. But in this case CA satisfies the restrictions.

Problem 24
Hint Put x-y=r, y-z=s and factorize.
Solution Put x-y=r, y-z=s. Then z-x = -(r+s), and $(x-y)^5 + (y-z)^5 + (z-x)^5 = r^5 + s^5 - (r+s)^5 = -5r^4s - 10r^3s^2 - 10r^2s^3 - 5rs^4 = -5rs(r+s)(r^2 + rs + s^2)$.

Problem 25
Hint Consider the shape of the curve if we plot the values a_r against r.
Solution The essential point is that if we plot the values a_r against r, then the curve formed by joining the points is cup shaped. Its two endpoints are on the axis, so the other points cannot be above it.There are many ways of turning this insight into a formal proof. Barry Paul's was neater than mine: $a_{r+1}-a_r >= a_r-a_{r-1}$. Hence (easy induction) if $a_s - a_{s-1}> 0$, then $a_n > a_s$. Take a_s to be the first positive, then certainly $a_s > a_{s-1}$, so $a_n > 0$. Contradiction.

Problem 26
Hint Consider the intersection of the biggest row and the biggest column.
Solution Induction on m+n. Trivial for m+n=2.
Let x be the largest number in the two given sets. Suppose it is a row total; let y be the largest column total. If y<x, then replace x by x-y in the set of row totals and remove y from the col totals. By induction find <=m+n-2 positive numbers in an m x (n-1) array with the new totals. Adding a col empty except for y in the row totalling x-y gives the required original set.
If y=x, then drop x from the row totals and y from the col totals and argue as before.
If x was a col total we interchange rows and cols in the argument.

Problem 27
Hint 3 points determine a circle.
Solution Let the circles be a, b, c, d, e. Let A be a point common to b, c, d, e, let B be a point common to a, c, d, e and so on. If any two of A, B, C, D, E coincide then the coincident point is on all 5 circles. Suppose they are all distinct. Then A, B, C are on d and e. Hence d and e coincide (3 points determine a circle). Hence D is on all 5 circles.

Problem 28
Hint The bottom 4 played 6 games amongst themselves.
Solution The bottom 4 played 6 games amongst themselves, so their scores must total at least 6. Hence the number 2 player scored at least 6. The maximum score possible is 7, so if the number 2 player scored more than 6, then he must have scored 6 1/2 and the top player 7. But then the top player must have won all his games, and hence the number 2 player lost at least one game and could not have scored 6 1/2. Hence the number 2 player scored exactly 6, and the bottom 4 players lost all their games with the top 4 players. In particular, the number 3 player won against the number 7 player.

Problem 29
Hint
(a) Show the point of intersection bisects each diagonal.
(b) Each pair of diagonals creates a pair of triangles of equal area.
Solution
(a) Let the quadrilateral be ABCD and let the diagonals AC, BD meet at E. Then area ABC = AC.EB.*sin* CEB/2, and area ADC = AC.ED.*sin* CEB/2, so E is the midpoint of BD. Similarly, it is the midpoint of AC. Hence the triangles AEB and CED are congruent, so angle CDE = angle ABE, and hence AB is parallel to CD. Similarly, AD is parallel to BC.
(b) Let the hexagon be ABCDEF. Let BE, CF meet at J, let AD, CF meet at K, and let AD, BE meet at L. Let AK=a, BJ=b, CJ=c, DL=d, EL=e, FK=f. Also let KL=x, JL=y and JK=z. Consider the pair of diagonals AD, BE. They divide the hexagon into 4 parts: the triangles ALB and DLE, and the quadrilaterals AFEL and BCDL. Since area ALB + area AFEL = area DLE + area BCDL, and area ALB + area BCDL = area DLE + area AFEL, the two triangles must have the same area (add the two inequalities). But area ALB = 1/2 AL.BL.*sin* ALB, and area DLE = 1/2 DL.EL.*sin* DLE = 1/2 DL.EL.*sin* ALB, so AL.BL = DL.EL or de = (a+x)(b+y). Similarly, considering the other two paris of diagonals, we get bc = (e+y)(f+z) and af = (c+z)(d+x). Multiplying the three inequalities gives: abcdef = (a+f)(b+y)(c+z)(d+x)(e+y)(f+z). But x, y, z are non-negative, so they must be zero and hence the three diagonals pass through a common point.

Problem 30
Hint Any common factor also divides 2mn.
Solution If d divides m+n and m^2+n^2, then it also divides $(m+n)^2 - (m^2+n^2) = 2mn$ and hence also $2m(m+n) - 2mn = 2m^2$ and $2n(m+n) - 2mn = 2n^2$. But m and n are relatively prime, so m^2 and n^2 are also. Hence d must divide 2.

Problem 31
Hint Take AY to be a diameter of c
Solution (a) Take Y on the circle so that angle ABY=90. Then AY is a diameter and so angle AMY=90. Take X as the midpoint of BY. Then triangles BXK and BYM are similar, so XK is parallel to YM. Hence XK is perpendicular to AM, and so P is the intersection of XK and AM. In other words, KP always passes through X.
(b) P must lie on the circle diameter AX, and indeed all such points can be obtained (given a point P on the circle, take M as the intersection of AP and the original circle). So the locus of P is the circle diameter AX.

Problem 32
Hint The center is the extreme point
Solution Answer: 2/3.

Let O be the center of ABC. Let AO meet BC at D, let BO meet CA at E, and let CO meet AB at F. Given any point X inside ABC, it lies in one of the quadrilaterals AEOF, CDOE, BFOD. Without loss of generality, it lies in AEOF. Take the line through X parallel to BC. It meets AB in P and AC in Q. Then PQ is shorter than the parallel line MON with M on AB and N on AC, which has length 2/3. If we twist the segment PXQ so that it continues to pass through X, and P remains on AB and Q on AC, then its length will change continuously. Eventually, one end will reach a vertex, whilst the other will be on the opposite side and hence the length of the segment will be at least that of an altitude, which is greater than 2/3. So at some intermediate position its length will be 2/3.
To show that no value smaller than 2/3 is possible, it is sufficient to show that any segment POQ with P and Q on the sides of the triangle has length at least 2/3. Take P on MB and Q on AN with P, O, Q collinear. Then PQ cos POM = MN - QN cospi/3 + PM cos pi/3. But PM>QN (using the sine rule, PM = OM sinPOM/sinOPM and QN = ON sinQON/sinOQN, but OM=ON, angle POM = angle QON, and angle OQN = angle OPM + pi/3 > angle OPM), and hence PQ > MN sec POM > MN.

Problem 33
Hint (a) How many dominoes straddle two adjoining columns? (b) No.
Solution
(a) We say a domino bridges two columns if half the domino is in each column. We show that for 0<n<6 the number of dominoes bridging columns n and n+1 must be at least 2 and even.
(b) Consider first n=1. There cannot be 3 dominoes entirely in column 1, or it would be separately tiled. So there must be at least one domino bridging columns 1 and 2. The number must be even, because it must equal the number of squares in column 1 (even) less twice the number of dominoes (entirely) in column 1.
Now suppose it is true for n<5 and consider column n+1. There must be at least one domino bridging columns n+1 and n+2, or columns 1 thru n+1 would be separately tiled. The number must be even, because it must equal the number of squares in column n+1 (even) less the number bridging n and n+1 (even) less twice the number entirely in the column. So in total there are at least 5 x 2 = 10 dominoes bridging columns. By the same argument there are at least another 10 bridging rows, but there are only 18 dominoes in total.

(b) No. For example:

```
1 2 3 3 1 1 2 2
1 2 1 2 2 3 3 1
3 3 1 3 1 2 4 1
1 2 2 3 1 2 4 3
1 3 3 2 2 1 2 3
3 2 1 1 4 1 2 1
3 2 3 2 4 3 3 1
1 1 3 2 1 1 2 2
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Problem 34
Hint Find a collection of n(n+1)/2 subsets with strictly increasing sums.
Solution Assume $a_1 < a_2 < ... < a_n$. We have the following collection of increasing sums:

$a_1 < a_2 < ... < a_n$ n sums
$a_1+a_n < a_2+a_n < ... < a_{n-1}+a_n$ n-1 sums
$a_1+a_{n-1}+a_n < a_2+a_{n-1}+a_n < ... < a_{n-2}+a_{n-1}+a_n$ n-2 sums

...

$a_1 + a_2 + ... + a_n$ 1 sum

A total of 1+2+ ... +n = n(n+1)/2.

Problem 35
Hint Use trigonometry, or show that DE = AC.
Solution The idea is to find an expression for the perpendicular distance h from D to AB. Let γ be the angle ACB, α half the angle CAB, and β half the angle ABC. We have h = AP sin α. Using the sine rule on APC, we have AP = AC $\sin(\gamma+\beta)/\sin(\alpha+\beta)$, so h = AC sin α $\sin(\gamma+\beta)/\sin(\alpha+\beta)$. Similarly, the perpendicular distance k from E to AB is BC sin β $\sin(\gamma+\alpha)/\sin(\alpha+\beta)$.
We also have that AC/BC = $\sin 2\beta/\sin 2\alpha$, and hence h/k = $\sin 2\beta \sin \alpha \sin(\gamma+\beta)/(\sin 2\alpha \sin \beta \sin(\gamma+\alpha))$. Using the fact that $\sin(\gamma+\beta) = \sin(2\alpha+\beta)$, and the expression for $\sin 2\theta$, we get h/k = $(\sin(2\alpha+2\beta) + \sin 2\alpha)/(\sin(2\alpha+2\beta) + \sin 2\beta)$ and hence h = k iff the triangle is isosceles. For some reason the geometric solution took me longer to find. Let ED meet BC at X. Then XCD and XBE are isosceles, so BC = BX + XC = DX + XE = DE. Similarly, AC = DE. Hence AC = BC.

Problem 36
Hint Let the square be a^2 and the difference d. Consider a^2+nd.
Solution Let the square be a^2 and the difference d, so that all numbers of the form a^2+nd belong to the arithmetic progression (for n a natural number). Take n to be 2ar + dr^2, then $a^2+nd = (a+dr)^2$.

Problem 37
Hint How many vertices are labeled 0?
Solution 10 x 5 > 45, so some digit i_0 must appear less than 5 times. But each occurrence can give at most 2 edges i_0, j, so there are at most 8 edges i_0, j, which is one too few.

Problem 38
Hint Set x = 1/2.
Solution Comparing coefficients of x^{20}, we must have $a = (2^{20} - 1)^{1/20}$ (note that we allow either the positive or the negative root).
Set x=1/2. Then we must have $(ax + b)^{20} = 0 = (x^2+px+q)^{10}$, and hence ax + b =0 and x^2+px+q = 0. So b = -a/2, and 1/4 + p/2 + q = 0.
Set x=0. Then we get $q^{10} = 1 - b^{20} = 1/2^{20}$, so q = 1/4 or -1/4, and p= -1 or 0 respectively. Comparing the coefficients of x^{19}, we must have p = -1 and q = 1/4. So, if there is a solution, then it must be: $a = (2^{20} - 1)^{1/20}$, b = -a/2, p = -1, q = 1/4. This is indeed a solution because with these values, the lhs = $2^{20}(x - 1/2)^{20} - (x - 1/2)^{20}a^{20} = (x - 1/2)^{20} = (x^2 - x + 1/4)^{10}$ = rhs.

Problem 39
Hint What is the sum of the new points added at step n?
Solution Answer: 2.3^{n-1}.
True for n=1. The new points added at step n+1 have twice the sum of the points after step n, because each old point contributes to two new points. hence the total after step n+1 is three times the total after step n.

Problem 40

Hint Take the circle with base as chord and the two sides as tangents.
Solution Let the triangle be ABC, with AB=AC. Take the circle through B and C which has AB and AC as tangents. The required locus is the arc BC.
Suppose P lies on the arc. Let the perpendiculars from P meet BC in L, AB in N and AC in M. Join PB and PC. The triangles PNB and PLC are similar (PNB and PLC are both 90, and NBP = LCP because NB is tangent to the circle). Hence PN/PL = PB/PC.
Similarly, triangles PMC and PLB are similar and hence PM/PL = PC/PB. Multiplying gives the required result PL^2 = PM.PN.
If P is inside the circle and not on it, take P' as the intersection of the line AP and the arc. We have PL<P'L, but PM>P'M and PN>P'N, hence PL^2<PM.PN. Similarly, if P is outside the circle and not on it, then PL^2>PM.PN.

Problem 41
Hint Consider the area.
Solution Let k be twice the area of the triangle. Then k>=BC^2, k>=AC^2 and k<=AC.BC, with equality in the last case only if AC is perpendicular to BC. Hence AC and BC have equal lengths and are perpendicular. So the angles are 90, 45, 45.

Problem 42
Hint m and m+1 must be powers separately.
Solution m and m+1 have no common divisors, so each must separately be an nth power. But the difference betwee the two nth powers is greater than 1 (for n>1).

Problem 43
Hint 9n+r reduces to r for 1<=r<=9.
Solution Taking digit sums repeatedly gives the remainder after dividing the number by 9, or 9 if the number is exactly divisible by 9. 10^9 - 1 = 9n, and for any r>=0 the nine consecutive numbers 9r+1, 9r+2, ... , 9r+9 include just one number giving remainder 1 and one number giving remainder 2. Hence the numbers up to 10^9 - 1 give equal numbers of 1s and 2s. 10^9 itself gives 1, so there is just one more of the 1s than the 2s.

Problem 44
Hint Consider what happens to the smallest value.
Solution Let the smallest value be s and suppose it occurs m times (with m<n). Then the values in the next stage are all at least s, and at most m-1 equal s. So after at most m iterations the smallest value is increased.
We can never reach a stage where all the values are equal, because if $(a_1+a_2)/2 = (a_2+a_3)/2 = ... = (a_{n-1}+a_n)/2 = (a_n+a_1)/2$, then $a_1+a_2 = a_2+a_3$ and hence $a_1 = a_3$. Similarly, $a_3 = a_5$, and so $a_1 = a_3 = a_5 = ... = a_n$ (n odd). Similarly, $a_2 = a_4 = ... = a_{n-1}$. But we also have $a_n + a_1 = a_1 + a_2$ and so $a_2 = a_n$, so that all a_i are equal. In other words, if all the values are equal at a particular stage, then they must have been equal at the previous stage, and hence at every stage.
Thus if the values do not start out all equal, then the smallest value increases indefinitely. But that is impossible, because the sum of the values is the same at each stage, and hence the smallest value can never exceed $(a_1 + ... + a_n)/n$.
Note that for n even the argument breaks down because a set of unequal numbers can iterate into a set of equal numbers. For example: 1, 3, 1, 3, ... , 1, 3.

Problem 45
Hint Extend three sides to form an equilateral triangle.

Solution (a) Extend AB, CD, EF. We get an equilateral triangle with sides AF + AB + BC, BC + CD + DE, ED + EF + FA. Hence AB - DE = CD - FA = EF - BC, as required.
(b) Take an equilateral triangle with sides s, t, u lengths $a_2 + a_3 + a_4$, $a_4 + a_5 + a_6$, and $a_6 + a_1 + a_2$ respectively. Construct BC length a_2 parallel to t with B on u and C on s. Construct DE length a_4 parallel to u with D on s and E on t. Construct FA length a_6 parallel to s with F on t and A on u. Then ABCDEF is the required hexagon, with AB = a_1, BC = a_2 etc.

Problem 46
Hint Consider first the case of 2 square roots.
Solution Let s_1 = sqrt(x), s_2 = sqrt(x + s_1), s_3 = sqrt(x + s_2) and so on. So the equation given is y = s_{1998}. We show first that all s_n must be integral for 1 <= n <= 1998. y is integral, so s_{1998} is integral. Now suppose s_n is integral. Then $s_{n-1} = s_n^2 - x$ is integral, proving the claim.
So in particular s_1 and s_2 are integers and $s_2^2 = s_1^2 + s_1$. But if $s_1 > 0$, then $s_1^2 < s_1^2 + s_1 < (s_1 + 1)^2$, which is impossible. Similarly $s_1 < 0$ is impossible. So the only possible solution is $s_1 = 0$ and hence x = 0 and y = 0.

Problem 47
Hint Let the diagonals meet at O. Look for triangles similar to OAA'.
Solution Let the diagonals meet at O. Then CC'O is similar to AA'O (because CC'O = AA'O = 90, and COC', AOA' are opposite angles), so A'O/C'O = AO/CO. Similarly, B'O/D'O = BO/DO. AA'O is also similar to BB'O, so A'O/B'O = AO/BO. Thus OA':OB':OC':OD' = OA:OB:OC:OD. Hence triangles OA'B' and OAB are similar. Likewise OB'C' and OBC, OC'D' and OCD, and OD'A' and ODA. Hence result.

Problem 48
Hint Suppose n is composite.
Solution Answer: n = 4 or prime.
If n = rs, with 1 < r < s, then r < s < n, and hence rsn = n^2 divides n!. Similarly, if n = r^2 with r > 2, then r < 2r < n, and hence n^2 divides n!. This covers all possibilities except n = 4 or n = prime, and it is easy to see that in these cases n^2 does not divide n!.

Problem 49
Hint Partition the vertices into two skewed lattices.
Solution

```
                                      1      2
    ./ \./ \.           directions    |     /
    |   |   |                         *     *     *
  ./°\./°\./°\.                                    \
  |   |   |   |                                     3
./°\./°\./°\./
|   |   |   |
*\./°\./°\./°
```

Suppose vertex A is that marked * at the bottom left. Without loss of generality, B is in a 60 degree sector as shown. Assume the edges have unit length. The vertices can be partitioned into two sets (marked ° and . in the diagram).
Each set forms a skewed lattice with axes at 60 degrees. Any path must alternate between the two lattices. If B is on the same lattice as A, then we can give B coordinates (m,n) relative to A and the shortest path from A to B must move m units east and n units east of north. The shortest path between a lattice point and the next lattice point east is evidently one edge in direction 3 followed by one edge in direction 2. Similarly, the shortest path between a lattice

point and the next lattice point east of north is one edge in direction 1, followed by one edge in direction 2. So a shortest path from A to B must have m+n edges in direction 2.
B is a distance √3(m+n/2) east of A and a distance 3n/2 north of A, so $AB^2 = (3m^2+3mn+3n^2) < (4m^2+8mn+4n^2) = 4(m+n)^2$. So in this case the bug must travel more than AB/2 in direction 2.
Now suppose B is on the other lattice. Let C be the lattice point immediately north of A and D the lattice point in direction 3 from A. Then a shortest path from A to B must either be A to C and then a shortest path from C to B, or A to D and then a shortest path from D to B. Take B to have coordinates (m, n) relative to C or D.
In the first case, $AB^2 = (\sqrt{3}(m+n/2))^2 + (3n/2 + 1)^2 = (3m^2 + 3mn + 3n^2) + 3n + 1$ and a shortest path has m+n units in direction 2. But $4(m + n)^2 > (3m^2 + 3mn + 3n^2) + 3n + 1$, if $m^2 + n^2 + 5mn > 3n + 1$, which is true for m, n at least 1. If m=0 and n=1, then a shortest path has 2 units in direction 1 and AB = √7 < 4. If m=1 and n=0, then AB=2 and a shortest path has 1 unit in each direction. So in this case (the only one so far) we have equality.
It remains to consider the case where the path starts out towards D. In this case $AB^2 = (\sqrt{3}(m+n/2) + \sqrt{3}/2)^2 + (3n/2 - 1/2)^2 = (3m^2+3mn+3n^2) + 3m + 1$ and a path has m + n units in direction 2. But $4(m + n)^2 > (3m^2 + 3mn + 3n^2) + 3m + 1$ for $m^2+n^2 + 5mn > 3m + 1$, which is true for m, n at least 1. If m=1, n=0, then a shortest path has 2 units in direction 3 and AB = √7 < 4. Finally, if m=0 and n=1, then a shortest path has 1 unit in each direction and AB = 2.
Thus the answer to the final question is 3, because the only cases where the bug travels exactly AB/2 in one direction are where it goes to the opposite vertex of a hexagon it is on.

Problem 50
Hint If AB touches the circle at W and AD at Z, then AO bisects angle ZOW.
Solution Let AB touch the circle at W, BC at X, CD at Y, and DA at Z. Then AO bisects angle ZOW and BO bisects angle XOW. So angle AOB is half angle ZOX. Similarly angle COD is half angle XOZ and hence angle AOB + angle COD equals 180.

Problem 51
Hint Find something else b - k must divide.
Solution We have $k^n - a = b^n - a$ (mod b - k). Hence $b^n - a = 0$ (mod b- k) for every k not equal to b. But if b^n does not equal a, then by taking $k - b > b^n - a$ we could render the equation false.

Problem 52
Hint Look for a pattern for low n.
Solution Answer 2^{n-2}. a_1 must be in the numerator, and a_2 must be in the denominator, but the other symbols can be in either. This is easily proved by induction.

Problem 53
Hint Consider the cube faces and the cube volume.
Solution Answer: 5.
Tetrahedral faces are triangular, so each cube face requires at least two tetrahedral faces. So at least 12 tetrahedral faces are needed in all. At most three faces of a tetrahedron can be mutually orthogonal (and no two faces can be parallel), so at most 3 faces from each tetrahedron can contribute towards these 12. So we require at least 4 tetrahedra to provide the cube faces. But these tetrahedra each have volume at most 1/6 (1/3 x face area x 1, and face area is at most 1/2). So if we have only 4 tetrahedra in total then their total volume is less than the cube's volume. Contradiction. Hence we need at least 5 tetrahedra.

It can be done with 5: lop off 4 non-adjacent corners to leave a tetrahedron. More precisely, take the cube as ABCDA'B'C'D' with ABCD horizontal, A' directly under A, B' directly under B and so on. Then the five tetrahedra are AA'BD, CC'BC, DD'A'C', BB'A'C', BDA'C'.

Problem 54
Hint (a) 121. (b) Answer: 16,36,121,484. To find solutions with more than 2 digits, write the number as $(10m+n)10^r + s$.
Solution (a) This one must have slipped through: 121!
(b) Answer: 16,36,121,484. Suppose the number has more than 2 digits. Write it as $(10m + n)10^r + s$, where $1 <= m <= 9$, $0 <= n <= 9$, $0 <= s < 10^r$. Then we have $k(m.10^r + s) = (10m +n)10^r + s$, for some $k > 1$.
s does not contain the digits 0 or 5, so 5 does not divide s. Hence 5 divides k-1, and so k must be 6, 11, or 16 (if k was 21 or more, then the rhs would be negative). Since 25 does not divide k-1, we must have r=1 and s is a single digit.
We look at each possibility for k in turn. k = 6 gives no solutions. k = 11 gives about two dozen multiples of 11 from 121 to 891. By inspection the only squares are 121 and 484. k = 16 gives 192, which is not a square.
In addition, there is the possibility of 2 digit solutions, which I had overlooked. It is easiest to check each of the 2 digit squares, thus finding the additional solutions 16, 36.

Problem 55
Hint The in-radius times the perimeter is twice the area. Also, AB + CD = AD + BC (because the quadrilateral has an inscribed circle).
Solution A necessary and sufficient condition for ABCD to have an inscribed circle is AB + CD = BC + AD. So we have AB + CD = 2AD, which we use repeatedly. Extend DC to X so that BX is parallel to EC. Then DX = AB + CD = 2AD and the triangles DEC, AEB, DBX are similar. Let h be the perpendicular distance from AB to CD. The similar triangles give us the heights of DEC and AEB in terms of h.
$1/r_1$ = perimeter ABE/(2 area ABE) = (AB + 2EB)/(AB.height) = (AB + 2.BD.AB/(AB+CD))/(AB.h.AB/(AB+CD)) = 2(AD + BD)/(AB.h). Similarly, $1/r_3$ = 2(AD + BD)/(CD.h).
The area of AED = area ABD - area ABE = 1/2 AB.h.CD/(2AD), so $1/r_2 = 1/r_4$ = perimeter ADE/(2 area ADE) = (AD + BD)/(h.AB.CD/2AD), and $1/r_2 + 1/r_4$ = 2(AD + BD)/h 2AD/(AB.CD) = 2(AB + BD)/h (AB + CD)/(AB.CD) = $1/r_1 + 1/r_3$.

Problem 56
Hint Express twice the sum as $(x_1 + ... + x_n)^2 - (x_1^2 + ... + x_n^2)$.
Solution (a) Answer: -[n/2].
Let $A = (x_1 + ... + x_n)^2$, $B = x_1^2 + ... + x_n^2$. Then we must minimize A - B. For n even, we separately minimize A and maximize B by taking half the x's to be +1 and half to be -1. For n odd we can take [n/2] x's to be +1, [n/2] to be -1, and one to be 0. That minimizes A and gives B one less than its maximum. That is the best we can do if we fix A = 0, since A = 0 requires an even number of x's to be non-zero and hence at least one to be zero. If we do not minimize A, then since its value must be an integer, its value will be at least 1. In that case, even if B is maximized we will not get a lower total.
(b) Answer: -[n/2]. For n even, the same argument works. For n odd we can clearly get -[n/2], so it remains to prove that we cannot get a smaller sum. Suppose otherwise, so that x_i is a minimal sum with sum less than -[n/2]. Let $x_n = x$, then the sum is $x(x_1 + ... + x_{n-1})$ + sum of terms x_ix_j with $1 <= i, j < n$. But this is less than the sum for n-1, so $x(x_1 + ... + x_{n-1})$ must be negative, and since it is minimal we must have $|x| = 1$. But the same argument shows that all

the terms have modulus 1. We now have a contradiction since we know that the minimum in this case is -[n/2].

Problem 57
Hint The first player always wins. Let the cards be $a_1>a_2> ... >a_9$. The first player secures $a_1 + a_9$ or forces it on his opponent, whichever is better.
Solution The first player always wins.
Let the board be:

```
. F .
S . S
. F .
```

We call the squares marked F the F-squares, the squares marked S the S-squares, and the remaining squares the neutral squares. The first player wins if the sum of the two cards on the F-squares exceeds the sum of the two cards on the S-squares. We also call the first player F and the second player S.
Let the cards be $a_1>a_2> ... >a_9$. Let $t_1 = a_1 + a_9$, $t_2 = a_2 + a_8$, $t_3 = a_3 + a_7$, $t_4 = a_4 + a_6$.
If $t_1 > t_2$, or $t_1 = t_2 > t_3$, or $t_1 = t_2 = t_3 >= t_4$ (*), then F's strategy is to get a total of t_1 or better on the F-squares and to force S to a lower score on the S-squares. If (*) does not hold, then F's strategy is to force S to t_1 or lower, and to get a higher score.
If (*) holds, then F starts by playing a_1 to an F-square. S must play to the remaining F-square, otherwise F will play a_3 or better to it on his next move and win. So S must play a_9 to the remaining F-square, giving F a total of t_1.
Now if $t_1 > t_2$, then F forces S to t_2 or worse by playing a_8 to an S-square.
If $t_1 = t_2 > t_3$, then F forces S to t_3 or worse by playing a_2 to a neutral square. If S plays to an S-square, then he cannot do better than $a_3 + a_8$, which loses. So he plays a_8 to a neutral square. But now F plays a_3 to an S-square, and S cannot do better than t_3.
If $t_1 = t_2 = t_3 > t_4$, then F forces S to t_4 or worse. He starts by playing a_2 to a neutral square. If does not prevent F playing a_8 to an S-square on his next move, then he cannot do better than $a_3 + a_8$, which loses. So he must play a_8 to a neutral square. Now F plays a_3 to a neutral square. If S does not prevent F playing a_7 to an S-square on the following move, then he cannot do better than $a_4 + a_7$ which loses, so he plays a_7 to a neutral square. F now plays a_4 to an S-square. S cannot now do better than t_4, which loses. Finally, if $t_1 = t_2 = t_3 = t_4$, then F proceeds as in the last case except that at the end he plays a_4 to the last neutral square instead of to an S-square. S now gets $a_5 + a_6$ on the S-squares, which loses.
If (*) does not hold, then F starts by playing a_9 to an S-square. If S does not play to the other S-square, then F will play a_7 or a_8 there on his next move and S will lose. So S must play a_1 to the other square, and gets a total of t_1. F now plays to get t_2, t_3 or t_4 on the F-squares.
If $t_1 < t_2$, then F plays a_2 to an F-square and so gets at least t_2 and wins.
If $t_1 = t_2 < t_3$, then F plays a_8 to a neutral square. If S does not prevent F playing $a_2$to an F-square on his next move, then F will get at least $a_2 + a_7$ and win. So S must play a_2 to a neutral square. Now F plays a_3 to an F-square and so gets at least t_3 on the F-squares and wins.
Finally, if $t_1 = t_2 = t_3 < t_4$, then F plays as in the previous case, except that at the end he plays a_7 to a neutral square instead of a_3 to an F-square. S must prevent F playing a_3 to an F-square the following move, or F gets at least $a_3 + a_6$ and wins. So S plays a_3 to a neutral square. F now plays a_4 to an F square and so must get at least t_4, which wins.

Problem 58
Hint Use the angle bisector result (if K lies on BC and AK bisects angle BAC, then BK/CK = AB/AC).

Solution ZY bisects the angle AYB, so AD/BD = AY/BY. Similarly, XY bisects angle BYC, so CE/BE = CY/BY. But AY = CY. Hence AD/BD = CE/BE. Hence triangles BDE and BAC are similar and DE is parallel to AC.
Let BY intersect AC at W and AX at I. I is the in-center. AI bisects angle BAW, so WI/IB = AW/AB. Now consider the triangles AYW, BYA. Clearly angle AYW = angle BYA. Also angle WAY = angle CAY = angle ABY. Hence the triangles are similar and AW/AY = AB/BY. So AW/AB = AY/BY. Hence WI/IB = AY/BY = AD/BD. So triangles BDI and BAW are similar and DI is parallel to AW and hence to DE. So DE passes through I.

Problem 59
Hint 13 divides 1001.
Solution The total is made up of numbers of the form abcabc, and pairs of numbers abcxyz, xyzabc. The former is abc.1001 and the sum of the pair is 1001(abc + xyz). So the total is divisible by 1001 and hence by 13.

Problem 60
Hint v/8 is not best possible. Indeed, the minimum speed for the boat to reach the island unilluminated and its optimum strategy are not obvious. However, we do not need to go into such questions. Just look for a simple argument that suffices.
Solution Let the lighthouse be at L. Take time t = 0 at the moment the boat starts its run, so that at t = 0 it is at S a distance d from L, and thereafter it is at a distance less than d. Take A and B a distance d from L so that ALBS is a semicircle with diameter AB and S the midpoint of the arc AB. During the period to t = 2.5 pi.d/v the boat has traveled a distance less than d, so it cannot reach AB. But it is a distance less than d from L, so it must be inside the semicircle. But during this period the beam sweeps across from LA to LB and so it must illuminate the boat.

Problem 61
Hint 100 - 1 is not even.
Solution Every time a person is on duty he is paired with two other people, so if the arrangement were possible the number of pairs involving a particular person would have to be even. But it is 99.

Problem 62
Hint Use r.p = 2.area.
Solution Let BC be the side parallel to XY, h the length of the altitude from A, and r the radius of the in-circle. Then XY/BC = (h - 2r)/h. But r.p = h.BC. So XY = (p - 2BC)BC/p = $(p^2/8 - 2(BC - p/4)^2)/p$. So the maximum occurs when BC = p/4 and has value p/8.

Problem 63
Hint First show $x_{ii} = 0$, then that $x_{ij} + x_{ji} = 0$.
Solution Taking i = j = k, we have that $x_{ii} = 0$. Now taking j=k, we have that $x_{ij} = - x_{ji}$. Define $a_i = x_{i1}$. Then we have $x_{i1} + x_{1j} + x_{ji} = 0$. Hence $x_{ij} = a_i - a_j$.

Problem 64
Hint Yes. Place points to block rectangles with side <x. Any unblocked rectangle must have a long side > 1/x. Block all but the slimmest of these with another set of points. Finally block the slimmest possible.
Solution Yes. Place a grid of 900 points in 30 equally spaced rows and columns, so that each point is a distance 15/31 from its nearest neighbours (or 15/31 from the edge). This blocks all

rectangles except those slimmer than 1/2. Those slimmer than 1/2 must have length at least 2, so we can block them with a smaller set of rows and columns containing more finely spaced points.
Label the rows 1-30. In each of the 7 rows 3, 7, 11, 15, 19, 23, 27 place an additional 31 points, so that each of these rows has 61 equally spaced points at a spacing of 15/62. Similarly for the columns. So in total we are placing an additional 2.7.31 = 434 points.
Any rectangle of length >2 must encounter one of these rows (or columns) and hence must have width less than 1/4. This blocks any rectangle except those with width < 1/4.
In each of the 3 rows 7, 15, 23 place an additional 62 points, so that each of these rows has 123 equally spaced points at a spacing of 15/124. Similarly for the columns. So in total we are placing an additional 2.3.62 = 372 points. Any rectangle of length >4 must encounter one of these rows (or columns) and hence must have width less than 1/8. This blocks any rectangle except those with width < 1/8 and hence length > 8.
In row 15 place an additional 124 points, so that it has a total of 247 equally spaced points at a spacing of 15/247. Similarly for column 15. This requires an additional 248 points. Any rectangle which can fit through these gaps has area at most 15 x 15/247 < 1. So we have blocked all rectangles with area 1 or more and used 900 + 434 + 372 + 248 = 1954 points.
Ilan Mayer, who seems to solve these problems effortlessly, came up with a neater arrangement of points. He used narrowly spaced points along widely spaced diagonals: (k/15, k/15) for k = 1,2,...,224; ((28*n+k)/15, k/15) for n = 1,2,...,7, k = 1,2,...,224-28*n; (k/15, (28*n+k)/15) for n = 1,2,...,7, k = 1,2,...,224-28*n. The diagonals are spaced 28/15 apart, so the biggest rectangle that can be fitted between two diagonals has sides 15/15 less epsilon and 15/15 less epsilon. For example, take the vertices as (14/15 + e, e), (29/15 - e, e), (14/15 + e, 15/15 - e), (29/15 - e, 15/15 - e). If one allows a rectangle to touch points (in other words if one took the rectangles to exclude their boundaries) then this does not work - many 15 x 1/15 rectangles will fit. But one can add an additional point on each of the 15 lines, keeping the points on each line evenly spaced. That blocks rectangles without boundary and still has only 1821 points.

Problem 65
Hint We may take all a_i to be in the range $0 < a_i < 1$. Show that for a maximal set all the a_i must be equal.
Solution We can take all a_i to lie in the range (0,1) and all b_i to be 0 or 1. The largest positive value of the sum of $(a_i - b_i)$ for any subset is achieved by taking the subset of those i for which $b_i = 0$. Similarly, the largest negative value is achieved by taking those i for which $b_i = 1$. So the worst subset will be one of those two.
If $a_i < a_j$, then we cannot have $b_i = 1$ and $b_j = 0$ if the set of b_is is to minimise the maximum sum, because swapping them would reduce the sum of a's with b = 0 and the sum of (1 - a)'s with b = 1. So if we order the a's so that $a_1 <= a_2 <= ... <= a_n$, then a best set of b's is $b_i = 0$ for i <= some k, and $b_i = 1$ for i > k. [If some of the a_i are equal, then we can find equally good sets of b's do not have this form, but we cannot get a lower maximum sum by departing from this form.]
Let $L_i = a_1 + a_2 + ... + a_i$, and $R_i = a_{i+1} + a_{i+2} + ... + a_n$. As we increase i the sums L_i increase and the sums R_i decrease, so for some k we must have $L_k < R_k$, $L_{k+1} >= R_{k+1}$. Either k or k+1 must correspond to the optimum choice of b's to minimise the maximum sum.
Now assume that the a's form a maximal set, in other words they are chosen so that the minimum is as large as possible. We show first that in this case $L_{k+1} = R_k$. Suppose $L_{k+1} < R_k$. Then we could increase each of $a_{k+1}, a_{k+2}, ... , a_n$ by epsilon. This would leave L_k unaffected, but slightly increase L_{k+1} and slightly reduce R_k. For small epsilon this does not change the value of k, but increases the smaller of L_{k+1} and R_k, thus increasing the minimum and

contradicting the maximality of the original a's. Similarly, if $L_{k+1} > R_k$, we could decrease each of $a_1, a_2, \dots, a_{k+1}$ by epsilon, thus slightly increasing R_k and reducing L_{k+1}.
Suppose not all of $a_1, a_2, \dots, a_{k+1}$ are equal. Take i so that $a_i < a_{i+1}$. Now increase each of $a_1, a_2, \dots, a_i$ by epsilon and reduce each of $a_{i+1}, a_{i+2}, \dots, a_{k+1}$ by epsilon', with epsilon and epsilon' sufficiently small that we do not upset the ordering or change the value of k, and with their relative sizes chosen so that L_{k+1} is increased. R_k is also increased, so we contradict the maximality of the a's. Hence all $a_1, a_2, \dots, a_{k+1}$ are equal. Similarly, we show that all of $a_{k+1}, \dots, a_n$ are equal. For if not we can increase slightly $a_{k+1}, \dots, a_j$ and reduce slightly $a_{j+1}, \dots, a_n$ to get a contradiction.
So we have established that all the a's must be equal. Suppose n is odd = 2m+1 and that all the a's equal x. Then for the optimum k we have (k+1)x = (2m+1-k)(1-x), hence k+1 = (2m+2)(1-x) and the maximum difference is (k+1)x = (2m+2)(1-x)x. This is maximised by taking x = 1/2, k = m, and is (m+1)/2 = (n+1)/4. If n is even = 2m, then for the optimum k we have (k+1)x = (2m-k)(1-x), so k+1 = (2m+1)(1-x), and the maximum difference is (k+1)x = (2m+1)(1-x)x. However, in this case we cannot take x = 1/2, because that would give k = m - 1/2 which is non-integral, so we take k = m-1 or m, both of which give a maximum difference of m(m+1)/(2m+1) = n(n+2)/(4n+4) < (n+1)/4.

Problem 66
Hint Each vertex except A and B has even degree, counting edges the number of times they were traversed.
Solution Disregard all edges except those used in the path from A to B, and for each of those let the multiplicity be the number of times it was traversed. Let the degree of a vertex be the sum of the multiplicities of its edges. The key is to notice that the degree of every vertex except A and B must be even. For as we traverse the path from A to B we increase the degree by 2 each time we pass through a vertex. But at the start of the path, as we leave A, we only increase its degree by 1. Similarly as we arrive at B for the last time.
Now construct a path from B as follows. Since B has odd degree it must have an edge of odd multiplicity. Suppose the edge connects B to C. Follow that edge and reduce its multiplicity by one, so that B's degree and C's degree are each reduced by one. Now C has odd degree, so it must have an edge of odd multiplicity. Repeat. Since there are only finitely many edges we must eventually be unable to continue the path. But the only way that can happen is if we reach A.

Problem 67
Hint (a) Each meeting involves 45 pairs. (b) A subcommittee has 10 pairs.
Solution (a) Each meeting involves 10.9/2 = 45 pairs. So after 40 meetings, there have been 1800 pairs. We are told that these are all distinct. But if there are N people on the committee, then there are only N(N-1)/2 pairs available. For N=60, this is only 1770.
(b) A subcommittee of 5 has 5.4/2 = 10 pairs. So 31 subcommittees have 310 pairs, and these are all distinct, since no two people are on more than one subcommittee. But a committee of 25 only has 25.24/2 = 300 pairs available.

Problem 68
Hint rs - r - s.
Solution Notice that 0 is good and all negative numbers are bad. Take c = rs - r - s. First c, is bad. For suppose otherwise: c = mr + ns. Then mr + ns = (s-1)r - s. Hence (s-1-m)r = (n+1)s, so r divides n+1. Say n+1=kr, and then s-1-m=ks, so m = (1-k)s - 1. But n+1 is positive, so k>=1, and hence m is negative. Contradiction.

If k is good, then c-k must be bad (otherwise c would be good). Suppose k is bad. Since r and s are relatively prime we can find integers a and b with ar + bs = 1 and hence integers m and n with mr + ns = k. Adding a multiple of sr - rs to both sides if necessary, this gives a pair m, n with mr + ns = k and m non-negative. Now take the pair with the smallest possible non-negative m. Then m <= s-1 (for otherwise m' = m-s, n' = n+r would be a pair with smaller non-negative m). Also n <= -1, otherwise k would be good. Now c - k = (s - 1 - m)r + (-n - 1)s and the coefficients s - 1 - m and -n - 1 are both non-negative, so c - k is good.
So exactly (rs - r - s + 1)/2 integers are bad.

Problem 69
Hint The missile moves on a semicircle.
Solution Answer: 18pi sec.
Let C be the position of the spy-plane at the moment the missile is fired. Let B be the point a quarter of the way around the circle from C (in the direction the spy-plane is moving). Then the missile moves along the semi-circle on diameter AB and hits the plane at B.
To see this take a point P on the quarter circle and let the line AX meet the semi-circle at Q. Let O be the center of the semicircle. The angle BOQ is twice the angle BAQ, so the arc BP is the same length as the arc BQ. Hence also the arc AQ is the same length as the arc CP.

Problem 70
Hint Consider the polyhedron as a sort of irregular tube between the two points. Then the tube always has at least three sides.
Solution If A and B are at the greatest distance, then they must be vertices. For suppose A is not a vertex. Then there is a segment XY entirely contained in the polyhedron with A as an interior point. But now at least one of angles BAX, BAY must be at least 90. Suppose it is BAX. Then BX is longer than BA. Contradiction.
Take a plane through A perpendicular to the line AB. Then the polyhedron must lie entirely on one side of the plane, for if Z lay on the opposite side to B, then BZ would be longer than BA. Now move the plane slightly towards B keeping it perpendicular to AB. The intersection of the plane and the polyhedron must be a small polygon. The polygon must have at least 3 vertices, each of which must lie on an edge of the polyhedron starting at A. Select three of these edges. As the plane is moved further towards B, the selected vertices may sometimes split into multiple vertices or they may sometimes coalesce. In the former case, just choose one of the daughter vertices. In the latter case, let O be the point of intersection of the plane and AB. Let O' be the point of intersection at the last coalescence (or A if there was none). Then we have three paths along edges, with no edges in common, each of which projects onto O'O and hence has length at least O'O. Now select one or more new vertices to replace any lost through coalescence and repeat.

Problem 71
Hint Fly in a spiral from one pole to the other, trapping the alien in an ever-reducing area.
Solution The spacecraft flies at a constant height, so that it can see a circular spot on the surface. It starts at the north pole and spirals down to the south pole, overlapping its previous track on each circuit. The alien cannot move fast enough to cross the track before the next circuit, so it is trapped inside a reducing area surrounding the south pole.
The value of 10 is not critical, so we do not have to optimise the details. Take the height above the surface to be half the radius. Then a diameter of the spot subtends an angle 2 $\cos^{-1}(1/1.5)$ at the center of the planet. $1/1.5 < 1/\sqrt{2}$, so the angle is more than 90 degrees. The critical case is evidently when the spacecraft is circling the equator. Using suitable units, we may take the radius of the planet to be 1 and the spaceship speed to be 1. Then the diameter of the spot is

pi/2. We take the overlap to be 2/3, so that each revolution the track advances pi/6. If the planet flew in a circle above the equator, the distance for a revolution would be 2pi 1.5 = 3pi. The helical distance must be less than 3pi + pi/6 = 19pi/6. So the alien can travel a distance 19pi/60 < 2/3 pi/2 and is thus trapped as claimed.

Problem 72
Hint There cannot be any loops of more than 2 soldiers, where each watches the next.
Solution The key is to notice that no loops of size greater than two are possible. For suppose we have $A_1, A_2, \dots, A_n$ with A_i watching A_{i+1} for $0 < i < n$, and A_n watching A_1. Then the distance $A_{i-1}A_i$ is greater than the distance A_iA_{i+1} for $1 < i < n$, and the distance A_1A_n is less than the distance A_1A_2. Hence the distance A_1A_n is less than the distance $A_{n-1}A_n$ and so A_{n-1} is closer to A_n than A_1. Contradiction.
Pick any soldier. Now pick the soldier he is watching, and so on. The total number of soldiers is finite so this process must terminate with some soldier watching his predecessor. If the process terminates after more than two soldiers have been picked, then the penultimate soldier is watched by more than one soldier. But in that case there must be another soldier who is unwatched, because the number of soldiers equals the number of soldiers watching.
If the process terminates after just two soldiers, then we have a pair of soldiers watching each other. Now repeat on the remaining soldiers. Either we find a soldier watched twice (in which case some other soldier must be unwatched) or all the soldiers pair off, except one, since the total number is odd. But that soldier must be unwatched.

Problem 73
Hint (a) Take P' so that the midpoint of PP' is the midpoint of AD. (b) Consider points P on AD.
Solution (a) Suppose the points lie in the order A, B, C, D. If P lies on AD, then the result is trivial, and we have equality if P lies outside the segment AD. So suppose P does not lie on AD.
Let M be the midpoint of AD. Take P' so that P, M, P' are collinear and PM = MP'. Then we wish to prove that PA + AP' > PB + BP'. Extend P'B to meet PA at Q. Then P'A + AQ > P'Q, so P'A + AP > P'Q + QP. But QP + BQ > PB, so QP + QP' > PB + PB'. Hence result.
(b) Let the foot of the perpendicular from B, C onto AD be X, Y respectively. Suppose that N, the midpoint of XY, is on the same side of M, the midpoint of AD, as D. Then take P to be a remote point on the line AD, the opposite side of A to D, so that A, D, M and N are all on the same side of the line PAD from P. Then PA + PD = 2PM < 2PN <= PB + PC. Contradiction. So we must have N coincide with M. But we still have PA + PD = 2PM = 2PN < PB + PC, unless both B and C are on the line AD. So we must have B and C on the line AD and AB = CD. It remains to show that B and C are between A and D. Take P = B. Then if C is not between A and D, we have PC > PD (or PA), contradiction.

Problem 74
Hint What is the smallest square greater than x^2?
Solution No. The smallest square greater than x^2 is $(x+1)^2$, so we must have y > 2x. Similarly x > 2y. Contradiction.

Problem 75
Hint Rearrange the back row, keeping each child in the front row in front of the same person. Now if two children in the front row are out of order, we can switch them.
Solution Rearrange the children in the back row into order, and rearrange the front row in the same way, so that each child stays in front of the same child in the back row. Denote heights in

the back row by a_i and heights in the front row by b_i. So we have $a_1 <= a_2 <= \dots <= a_n$, and $a_i > b_i$ for i = 1, 2, ... , n.
Now if i < j, but $b_i > b_j$, then we may swap b_i and b_j and still have each child taller than the child in front of him. For $b_i < a_i <= a_j$, and $b_j < b_i < a_i$. By repeated swaps we can get the front row into height order. [For example, identify the shortest child and swap him to the first position, then the next shortest and so on.]

Problem 76
Hint The number of shortest paths going a distance m in one direction and n in the other is (m + n)!/(m! n!).
Solution Let ABCD have n lattice points along the side AB. Then it has kn lattice points along the side AD. Let X be the first lattice point along AB after leaving A. A shortest path from X to C must involve a total of kn + n - 1 moves between lattice points, n - 1 in the direction AB and kn in the direction BC. Hence the total number of such paths is (kn + n - 1)!/((kn)! (n - 1)!). Similarly, the number of paths starting out along AD is (kn + n - 1)!/((kn - 1)! n!). Let m = (kn + n - 1)!/((kn - 1)! (n - 1)!). Then the number starting along AB is m/(kn) and the number starting along AD is m/n, which is k times larger, as required.

Problem 77
Hint We can pick b_n, then b_{n-1} and so on.
Solution We show that you can pick $b_n, b_{n-1}, \dots, b_r$ so that $s_r = b_na_n + b_{n-1}a_{n-1} + \dots + b_ra_r$ satisfies $0 <= s_r <= a_r$. Induction on r. Trivial for r = n. Suppose true for r. Then $- a_{r-1} <= s_r - a_{r-1} <= a_r - a_{r-1} <= a_{r-1}$. So with $b_{r-1} = -1$ we have $|s_{r-1}| <= a_{r-1}$. If necessary, we change the sign of all $b_n, b_{n-1}, \dots, b_{r-1}$ and obtain s_{r-1} as required. So the result is true for all r >= 1 and hence for r = 1.

Problem 78
Hint Show that the area allowed for the center of the circle is greater than zero.
Solution Draw a rectangle width A/P on the inside of each side. The rectangles at each vertex must overlap since the angle at the vertex is less than 180. The total area of the rectangles is A, so the area covered must be less than A. Hence we can find a point not in any of the rectangles. But this point must be a distance more than A/P from each side, so we can use it as the center of the required circle.

Problem 79
Hint Take any path from A to B and use induction to get disjoint paths from A to each vertex along it.
Solution Take any path from A to B. Suppose it is $A=A_0, A_1, \dots, A_n=B$. We show by induction on r that we can find two disjoint paths from A to A_r. If r = 1, then take any vertex C distinct from A and A_1. Take any path from A_1 to C which does not go through A. Now take any path from C to A which does not go through A_1. Joining these two paths together gives a path p from A to A_1 which does not involve the edge AA_1. Then p and the edge AA_1 are the required disjoint paths.
Suppose now we have two disjoint paths $A, B_1, B_2, \dots, B_s, A_r$ and $A, B_t, B_{t-1}, \dots, B_{s+1}, A_r$ and we wish to find two disjoint paths joining A and A_{r+1}. Take a path between A and A_{r+1} which does not include A_r. If it also avoids all of $B_1, \dots, B_t$, then we are home, because it is disjoint from the alternative path $A, B_1, B_2, \dots, B_s, A_r, A_{r+1}$. If not, let B_i be the first of the B's on the path as we move from A_{r+1} to A. This allows us to construct two disjoint paths from A to A_{r+1}. One path goes from A to B_i and then from B_i to A_{r+1}. The other path goes around the other way to A_r and then along the edge to A_{r+1}. [Explicitly, if i <= s, then the paths are $A, B_1, B_2, \dots, B_i$, ... (new path) ... A_{r+1} and $A, B_t, B_{t-1}, \dots, A_r, A_{r+1}$. If i > s, then the paths are $A, B_t, B_{t-1}, \dots, B_i$, ...

(new path) ... A_{r+1} and A, B_1, ... , B_s, A_r, A_{r+1}.] Hence, by induction, there are two disjoint paths from A to

Problem 80

Hint Consider the locus of H for which PH is shorter than the altitude from A. How far can H be from BC?

Solution Answer: the triangle DEF with FAE parallel to BC, DBF parallel to CA and DCE parallel to AB.

Let α be the angle between planes ABC and PBC. Let h be the perpendicular distance from H to the line BC, and let h_A be the perpendicular distance from A to the line BC. Then PH = h tan α, and the altitude from A to PBC is h_A sin α. Hence if PH is shorter than the altitude from A we require that $h < h_A \cos\alpha < h_A$. Similar arguments apply for B and C. So if PH is the shortest then H lies within triangle DEF.

If H does lie within DEF, then if we make α sufficiently small we will have $h < h_A \cos\alpha$ and hence PH will be shorter than the altitude from A. Similarly we can make PH sufficiently short that PH is less than the altitudes from B and C. Hence the inside of DEF is the required locus.

Problem 81

Hint If we use m circles then we can state a stronger result (the more circles the smaller the total diameter). Prove it by induction.

Solution If we have two circles diameters d and d', the distance between which is less than 1, then they are contained in a circle diameter d+d'+1. [If the line through the centers cuts the circles in A, B, A', B', then take a circle diameter AB'.] So start with 100 circles of diameter 1/1000 each. If any pair is a distance <= 1 apart, then replace them by a single circle, increasing the total diameter by 1. Repeat until all the circles are a distance > 1 apart. We must end up with at least one circle, so the total increase is at most 99. Hence the final total diameter is at most 99 1/10.

Problem 82

Hint Answer: 20πd√(αβ) kilometers per hour.

Obtain relations between the distance the plane moves in 1 sec, its height and the (horizontal) distance from A. Then use the fact that the angles α and β are small.

Solution Answer: 20πd√(αβ) kilometers per hour.

Let the plane be at height h and a (horizontal) distance y from A. Let the angle PAB be θ+α and the angle PBA be φ. After 1 second, the angle PAB is θ and the angle PBA is φ+β. We have immediately that:

$h/y = \tan(\theta+\alpha)$, $h/(d-y) = \tan\phi$, $h/(y+x) = \tan\theta$, $h/(d-y-x) = \tan(\phi+\beta)$.

Eliminating θ, we obtain: $h/y = (\tan\alpha + \tan\theta)/(1 - \tan\alpha\tan\theta) = (a(y+x) + h)/(y+x-ah)$, where $a = \tan\alpha$. Hence $x = a(h^2 + y^2)/(h - ay)$. Similarly, eliminating φ, we obtain $x = b(h^2 + (d-y)^2)/(h + (d-y)b)$.

At this point I do not see how to make further progress without approximating. But approximating seems reasonable, since α and β, are certainly small, at least when expressed in radians. For example, typical values might be 10,000 ft for h and more than 10 miles for y or d-y and 500 mph for the aircraft speed. That gives x = 0.14 miles, so x/y = 0.014 and x/h = 0.07. So, let us neglect a/h, b/h, a/y etc. Then we get the simplified expressions: $x = a(h^2 + y^2)/h = b(h^2 + (d-y)^2)/h$.

If a = b, then we quickly obtain y = d/2, h = d/2, x = ad. Assume a > b. Then we can solve for h, substitute back in and obtain an expression for x in terms of y. It is convenient to divide through by d and to write X = x/d, Y = y/d. Note that since we are assuming a > b, we require

$Y < 1/(1 + \sqrt{a/b})$. After some manipulation we obtain: $X = ab(1 - 2Y)/\sqrt{(a-b)(b(1-Y)^2 - aY^2)}$. Differentiating, we find that there is a minimum at Y = b/(a+b), which is in the allowed range, and that the minimum value of X is √(ab). By symmetry, we obtain the same result for a < b and we notice that it is also true for a = b. So in all cases we have that the minimum value of x is d√(ab).

We are assuming α and β are small, so we may take a = α, b = β. However, the question specified that α and β were measured in degrees, so to obtain the final answer we must convert, giving: x = d (π/180) √(αβ), and hence speed = 20πd√(αβ) kilometers per hour.

Problem 83

Hint Answer: 30.

Find a strategy for the 2nd player which gets at least this total and a strategy for the 1st player which gets at most this total.

Solution Answer: 30.

The second player can play the following strategy: (1) if the first player plays 2n-1 for 1 <= n <= 9, then he replies 2n with the opposite sign; (2) if the first player plays 2n for 1 <= n <= 9, then he replies 2n-1 with the opposite sign; (3) if the first player plays 19 or 20, then he plays the other with the same sign. This secures a score of at least 39 (from (3)) less 9 x 1 (from (1) and (2)). So he can ensure a score of at least 30.

The first player can play the following strategy: (1) he opens with 1; (2) if the second player plays 2n for 1 <= n <= 9, then he replies with 2n+1 with the opposite sign; (3) if the second player plays 2n+1 for 1 <= n <= 9, then he replies with 2n with the opposite sign; (4) if any of these replies are impossible, or if the second player plays 20, then he replies with the highest number available with the opposite sign. If the second player does not play 20 until the last move, then this strategy ensures a score of at most 1 (from (1)) + 9 x 1 (from (2) and (3)) + 20 = 30. Now suppose that the second player plays 20, a_1, a_2, ... , a_n (where 1 <= n <= 9) which require a reply under (4). The reason a_1 required a move under (4) was that a_1-1 or a_1+1 was the 1st player's response to 20. Similarly, the reason a_2 required a move under (4) was that a_2-1 or a_2+1 was the 1st player's response to a_2, and so on. Thus the increment to the absolute value from these moves is at most $|20-a_1+1| + |a_1-a_2+1| + ... + |a_{n-1}-a_n+1| + |a_n| = 20 + n$. The increment from the moves under (2) and (3) is (9 - n) x 1, and the increment from the move under (1) is 1. Hence the maximum absolute value is 30.

Since the 1st player has a strategy to do no worse than 30 and the 2nd player has a strategy to do no worse than 30, these strategies must actually be optimal.

Problem 84

Hint Use trigonometry. For (b) we need only AH <= BM and AH <= CD (plus the facts that the triangle is acute-angled and AH is the longest altitude).

Solution As usual let a, b, c be the lengths of BC, CA, AB respectively and let A, B, C denote the angles BAC, ABC, BCA respectively. We use trigonometry and try to express the quantities of interest in terms of a, b and C.

(a) Since AH is the longest altitude, BC must be the shortest side (use area = side x altitude/2). So $b^2 >= a^2$, and $c^2 >= a^2$. Using the formula $c^2 = a^2 + b^2 - 2ab\cos C$, we deduce that $b^2 >= 2ab\cos C$. Hence $2b^2 >= a^2 + 2ab\cos C$. After a little manipulation this gives: $a^2 + b^2 - 2ab\cos C >= 4/3\,(a^2 + b^2/4 - ab\cos C)$ or $c^2 >= 4/3\,BM^2$. But we are given that BM >= AH = b sin C, so $(b^2\sin^2 C)/c^2 <= 3/4$. But the sine formula gives sin B = (b sin C)/c, so $\sin^2 C <= 3/4$. The triangle is acute-angled, hence B <= 60 degrees.

(b) The angle bisector theorem gives AD/BD = b/a, hence AD/AB = b/(a+b), so AD = bc/(a+b). Hence, using the sine formula, CD/sin A = AD/(sin C/2). So CD = bc sin A/((a+b) sin C/2) = ba sin C/((a+b) sin C/2), using the sine formula again. But we are given that CD >= AH = b sin C, so a/((a+b) sin C/2) >= 1. But a is the shortest side, so a/(a+b) <= 1/2 and hence sin C/2 < 1/2. The triangle is acute-angled, so C/2 <= 30 degrees, and C <= 60 degrees. BC is the shortest side, so A is the smallest angle and hence A <= 60 degrees. Also since AH <= BM, B <= 60 degrees. But the angles sum to 180 degrees, so they must all be 60 degrees and hence the triangle is equilateral.

Problem 85
Hint (a) There cannot be any carries in the sum. (b) All carries must be 1.
Solution
(a) Let the digits of the original number be a_1, a_2, ... and the rearranged digits be b_1, b_2,... . Suppose that in the addition there is a carry, in other words $a_i+b_i > 9$ for some i. Take the largest such i. Then the resulting digit in that position cannot be a 9. Contradiction. So there cannot be any carries. Hence each pair $a_i+b_i= 9$. Let n be the total number of digits 0, 1, 2, 3, and 4 in the number. Then each of these must be paired with a digit 5, 6, 7, 8 or 9. So the total number of digits 5, 6, 7, 8 and 9 must also be n, and hence the number must have an even number of digits. But we are told that the answer and hence the original number has an odd number of digits.
(b) In the addition the carry can never be 2, because that would require the previous carry to be at least 2, and the first carry cannot be 2. So all carries are 0 or 1. If a carry is 1, then all subsequent carries must also be 1. If the first carry is 0, then the corresponding digits must be 0 and hence the original number is divisible by 10. If it is not, then all carries are 1 and hence after the first carry all the digit pairs sum to 9. But arguing as in (a), this means that there must be an even number of digits, excluding the last (where we have a digit sum 10), and hence an odd number of digits in the original number. But 10^{10} has an odd number of digits and hence the original number had an even number of digits. Contradiction.

Problem 86
Hint Relative the direction of one lamp the others must point in directions 90, 180 and 270. The choice must be arranged to avoid uncovered strips.
Solution Take a north direction, arbitary except that no points are aligned north-south or east-west. Take the two most northerly points. Point the lamp for the more easterly of these two in the direction SW (so that it covers directions S to W). Point the lamp for the other in the direction SE. For the other two points, point the lamp for the more easterly in the direction NW, and the lamp for the other in the direction NE.
Clearly the lamps cover all directions, the only possible problem is uncovered strips. However, the two lamps pointing N are below the two lamps pointing S, and the two lamps pointing E are west of the two lamps pointing W, so there are no uncovered strips.

Problem 87
Hint (a) No. (b) Yes.
Solution No. Each of the numbers 0, 1, 8, 9 can only be adjacent to 3, 4, 5 or 6. But they can only accomodate 3 numbers, not 4.
0, 3, 7, 10, 13, 9, 12, 8, 11, 6, 2, 5, 1, 4 is a solution for 13.
In passing, there are obviously no solutions for 4 or 5. There is just the one solution for 6 (given in the question). For 7 there are 5 solutions: 0, 3, 6, 1, 5, 2, 7, 4; 0, 3, 6, 1, 4, 7, 2, 5; 0, 3, 6, 2, 7, 4, 1, 5; 0, 3, 7, 4, 1, 6, 2, 5; 0, 4, 1, 6, 3, 7, 1, 5. For 8 there is the solution 0, 3, 7, 2, 6, 1, 5, 8, 4, and maybe others.

Problem 88

Hint If n 5^{1000} has a particular digit pattern, and m is the remainder when n is divided by 2^{1000}, what digit pattern does m 5^{1000} have?
Solution We first find a multiple of 5^{1000} which has no zeros in the last 1000 digits. Suppose that we have a multiple $n.5^{1000}$ whose last zero is in place r (treating the last place as place 0, the next to last as place 1 and so on). Then $n(10^r + 1)$ has the same digits in places 0 to r-1 and a non-zero digit in place r, and hence no zeros in places 0 to r. So repeating, we find a multiple $n.5^{1000}$ with no zeros in the last 1000 digits.
Now let m be the remainder when n is divided by 2^{1000}, so $n = k.2^{1000} + m$, and hence $m.5^{1000} = n.5^{1000} - k.10^{1000}$. So $m.5^{1000}$ has the same last 1000 digits as $n.5^{1000}$. But it has less than 1001 digits, and hence it has exactly 1000 digits and no zeros.

Problem 89
Hint Solutions are x,y = -1,-1; 0,-1; -1,0; 0,0; -6,2; or 5,2.
The square root of the rhs is about $y^2 + y/2$. So try values of x near this.
Solution The only solutions are x,y = -1,1-; 0,-1; -1,0; 0,0; -6,2; or 5,2.
$(y^2 + y/2 - 1/2)(y^2 + y/2 + 1/2) = y^4 + y^3 + 1/4\ y^2 - 1/4 < y^4 + y^3 + y^2 + y$ except for -1 <= y <= -1/3. Also $(y^2 + y/2)(y^2 + y/2 + 1) = y^4 + y^3 + 5/4\ y^2 + y/2$ which is greater than $y^4 + y^3 + y^2 + y$ unless 0 <= y <= 2.
But no integers are greater than $y^2 + y/2 - 1/2$ and less than $y^2 + y/2$. So the only possible solutions have y in the range -1 to 2. Checking these 4 cases, we find the solutions listed.

Problem 90
Hint Answer 2998. The sequence is completely determined by its first two elements. Find the length for various possible x_1, x_2 by induction
Solution Answer 2998.
The sequence is completely determined by its first two elements. If the largest element of the sequence is n, then it must occur as one of the first two elements. Because x_3 and x_4 are both smaller than the largest of the first two elements and hence all subsequent elements are too.
Let f(n, m) be the length of the sequence with $x_1 = n$, $x_2 = m$. It is straightforward to verify by induction that f(1,2n) = f(2n-1,2n) = 3n + 1, f(2n,1) = f(2n,2n-1) = 3n, f(2n,2n+1) = 3n + 3, f(1,2n+1) = f(2n+1,1) = 3n + 2, f(2n+1,2n) = 3n + 1. A rather more fiddly induction then shows that these are the best possible lengths. Hence the longest
sequence with no element more than 1998 is that starting 1, 1998 which has length 2998.

Problem 91
Hint (a) true, (b) and (c) false.
Solution (a) True. Black moves to one end of a main diagonal and then moves along the diagonal to the opposite end. Each of the 499 rooks is in some row. Since black moves through each row, every rook must change row. But each of the rooks is also in some column and so every rook must also change column. A rook cannot change row and column in the same move, so white must make at least 998 moves before black reaches the opposite end of the diagonal. But it cannot start until black is two moves from its starting position, because if it moves a rook into row (or column) one or two earlier, then black is checked or can move into check. So it has only 997 moves available, which is one too few.
(b) False. Suppose the contrary, that after move n, the king is always in check after its move. Let the corners of the board be A, B, C, D. After move n, white moves all its rooks inside a square side 23 at corner A. The king must now be in the 23 rows between A and B or in the 23 columns between A and D. Suppose the latter. Then white moves all its rooks inside a square side 23 at corner B. This should take 499 moves. However, it could take longer if black used his king to obstruct the move. The worst case would be 3 x 23 additional moves (the king can

only obstruct one row of 23 rooks, and each rook in the obstructed row could take 4 moves instead of one to reach its destination.). During this period the king must remain in the 23 rows from A to B or the 23 columns from A to D, since it must remain in check. Thus it cannot get to B by the completion of the process. In fact, it must be at least 999 - 46 (the total number of moves required) - (499 + 69) (the number of moves available) = 385 moves behind.
White now moves all the rooks inside a square side 23 at corner C. The king cannot cut across (or it will be unchecked). It must keep within 23 squares of the edge. So it ends up 770 moves behind (more in fact, since it cannot obstruct the move as effectively). Finally, white moves all the rooks inside a square side 23 at corner D. The king cannot get to the side CD by the time this process is completed. So there is then a lag of over two hundred moves before it can get back into check. Note that it does not help black to change direction. Whatever black does, white ends up with all the rooks at a corner and the king a long way from the two checked sides.
(c) False. This follows from (b). But we may also use a simpler argument. Take coordinates x = 1 to 1000, y = 1 to 1000. White gets its pieces onto (2,0), (4,0), ... , (998,0). If the king moves onto (2n,*), then white moves its rook from (2n,0) to (2n-1,0), leaving the king unchecked. If the king moves to (2n-1,*) or (2n+1,*), then white moves its rook back to (2n,0), leaving the king unchecked. If the king stays on the line (2n,*), then white fills in time by toggling one of its endmost rooks to an adjacent square (and the king remains unchecked). The only way the black king can escape this repeated unchecking is by moving up to the line y = 0. If it does so, then white transfers all its rooks to the line y = 1000 and repeats the process. The transfer takes 499 moves. It takes black 1000 moves to follow, so during the 501 moves before black catches up, the king is subject to repeated unchecking.

Problem 92
Hint Answer: 1.
Find the locus of the fourth vertex when the opposite vertex is kept fixed. [You should find it is a line segment.]
Solution Answer: 2 1/3.
Let the square be ABCD. Let the vertices of the rhombus be P on AB, Q on AD, and R on BC. We require the locus of the fourth vertex S of the rhombus. Suppose P is a distance x from B. We may take x <= 1/2, since the locus for x > 1/2 is just the reflection of the locus for x < 1/2. Then since PR is parallel to QS, S is a distance x from the line AD. Also, by continuity, as Q varies over AD (with P fixed a distance x from B), the locus of S is a line segment.
The two extreme positions for S occur when Q coincides with A and when R coincides with C. When Q coincides with A the rhombus has side 1-x. Hence $BR^2 = (1-x)^2 - x^2 = 1 - 2x$. In this case SR is parallel to AB, so the distance of S from AB is $\sqrt{(1-2x)}$. When R coincides with C, the rhombus has side $\sqrt{(1+x^2)}$, so $AQ^2 = 1 + x^2 - (1-x)^2 = 2x$. Hence the distance of S from AB is $1 + \sqrt{(2x)}$.
Thus the locus of S over all possible rhombi is the interior of a curvilinear quadrilateral with vertices MDNC, where M is the midpoint of AB and N is the reflection of M in CD. Moreover the curve from M to C is just the translate of the curve from D to N, for if we put y = 1/2 - x, then $\sqrt{(1-2x)}$ becomes $\sqrt{(2y)}$. Thus if L is the midpoint of CD, then the area in the MLC plus the area in DLN is just 1/2, and the total area of the curvilinear quadrilateral is 1.
However, the arrangement of the vertices discussed above is not the only one. The order of vertices above is PQSR. We could also have PQRS or PSQR. In either case QR is a side rather than a diagonal of the rhombus. We consider the case PQRS (the case PSQR is just the reflection in the line MN). As before it is convenient to keep P fixed, but this time we take x to be the distance AP. Take y to be the distance AQ.

As before we find that S must lie on a line parallel to BC a distance x from it (on the other side to AD). Again we find that for fixed P, the locus of S is a segment of this line. If we assume that AQ > BR, then the two extreme positions are (1) QR parallel to AB, giving S on the line AB, (2) Q at D, giving S a distance x from the line AB. So as x varies from 0 to 1 we get a right-angled triangle sides 1, 1 and $\sqrt{2}$ and area 1/2. However, we can also have BR > AQ. This gives points below the line AB. The extreme position is with R at C. Suppose QD = y. Then $1 + y^2 = x^2 + (1 - y)^2$, so $y = x^2/2$. This gives S a distance y below the line AB. This gives an additional area of 1/6 (by calculus - integrate $x^2/2$ from 0 to 1; I do not see how to do it without).
The triangle and the curvilinear triangle together form a curvilinear triangle area 1/2 + 1/6 = 2/3. There is an identical triangle formed by reflection in MN. Thus the total area is 1 + 2/3 + 2/3 = 2 1/3.
Thanks to Robert Hill and John Jones for pointing out that the original solution missed out the two triangles.

Problem 93
Hint The basic idea is to look at particular multiples of k and their reverse and then to form suitable linear combinations (which must be divisible by k). However, we have to look at relatively large multiples, because it is easy to find arbitarily large k such that all multiples of k up to k^2 have their reverse divisible by k. What we need is some multiple which is sure to have enough carries to spoil the reverse.
Solution Let r(m) denote the number obtained from m by reversing the digits.
We show first that k cannot be divisible by 2 or 5. It cannot be divisible by both, for then it ends in a zero and hence r(k) < k and so is not divisible by k (contradiction). So if 5 divides k, then the last digit of k must be 5. Since r(k) is divisible by 5 its last digit must also be 5, so the first digit of k is 5. But now 3k has first digit 1 (3.5 > 10 and 3.6 < 20), so r(3k) has last digit 1 and cannot be divisible by 5. Contradiction. If 2 divides k, then every multiple of k must be even. So the last digit of r(k) must be even and hence the first digit of k must be 2, 4, 6, or 8. If 2, then 5k has first digit 1, so r(2k) is odd. Contradiction. Similarly, if the first digit is 4, 3k has first digit 1; if 6, then 5k has first digit 3; if 8, then 2k has first digit 1. Contradiction. So k is not divisible by 2 or 5. Suppose $k = 10^n a_n + ... + a_0$. k divides r(k), so $a_0 >= 1$. Hence $(10^{n+1} - 1)k = 10^{2n+1} a_n + ... + 10^{n+1} a_0 - (10^n a_n + ... + a_0) = 10^{2n+1} a_n + ... + 10^{n+1}(a_0-1) + 10^n c_n + ... + 10 c_1 + (c_0+1)$, where $c_i = 9 - a_i$. The reverse of this, $10^{2n+1}(c_0+1) + 10^{2n} c_1 + ... + 10^{n+1} c_n + 10^n(a_0-1) + ... + a_n$, is also divisible by k. So is the reverse of k, $10^n a_0 + ... + a_n$ and hence also their difference: $10^n(10^{n+1}(c_0+1) + 10^n c_1 + ... + 10 c_n - 1)$. k has no factors 2 or 5, so k must divide $10^{n+1}(c_0+1) + 10^n c_1 + ... + 10 c_n - 1$. Adding 10k, we find that k also divides $10^{n+2} + 10^n 9 + ... + 10.9 - 1 = 1099...989$ (n - 2 consecutive 9s) $= 11(10^{n+1} - 1)$. We can now carry out exactly the same argument starting with $(10^{n+2} - 1)k$. This leads to k dividing $10^{n+2}(c_0+1) + ... + 10^2 c_0 + 10.9 - 1$ and hence also $10^{n+3} + 10^{n+1} 9 + ... + 10^2 9 + 10.8 + 9 = 11(10^{n+2} - 1)$. Subtracting 10 times this from the previous number we conclude that k must divide $11(10^{n+1} - 1) - 11(10^{n+1} - 10) = 99$. Finally, we note that any factor of 99 has the required property. For 3 and 9 divide a number if and only if they divide its digit sum. So if m is divisible by 3 or 9, then the number formed by any rearrangement of its digits is also divisble by 3 or 9. m is divisible by 11 if and only if the difference between the sums of alternate digits is divisible by 11, so if m is divisible by 11, then so is its reverse.

Problem 94
Hint Extend the sides to form two rectangles. Find the rectangle sides in terms of the octagon sides.

Solution Extend the sides to form two rectangles. Let the sides of the octagon have length a, b, c, d, e, f, g, h. Then we can find the rectangle sides. For example, one of the rectangles has opposite sides $a + (b + h)/\sqrt{2}$ and $e + (d + f)/\sqrt{2}$. Hence either a = e or $\sqrt{2} = (b + h - d - f)/(a - e)$. The root is irrational, so we must have a = e. Similarly for the other pairs of opposite sides.

Problem 95
Hint 17^{14} is far bigger (more than a factor 5).
Solution $17^2 = 289 > 9.31$. So $17^{14} > 9^7\, 31^7$. But $3^7 = 2187 > 31^2$. Hence $17^{14} > 31^{11}$.

Problem 96
Hint How many grid lines does it cross?
Solution Take compass directions aligned with the grid. Let N, E, S, W be the most northerly, easterly, southerly and westerly points on the circle. The arc from N to E must cross 100 north-south grid lines and 100 east-west grid lines. Each time it crosses a grid line it changes square (and it never crosses two grid lines at once, because it does not pass through any lattice points), so the arc N to E must pass through 200 in addition to the starting square. Similarly for the other 4 arcs. So the circle passes through a total of 800 squares (we count the starting square in the last 200).

Problem 97
Hint Show that the number of students speaking any two given languages exceeds the number of students speaking the third language by the same amount.
Solution Let EF denote the number of students speaking English and French. Similarly define ES, FS, E, F, S, EFS. Then ES + EF + E + EFS = 50, EF + FS + F + EFS = 50. Subtracting: ES - F = FS - E. Similarly, ES - F = EF - S.
Pair off members of FS with members of E. Similarly, members of ES with F,and members of EF with S. The resulting pairs have one person speaking each language. If ES = F, then the only remaining students are those in EFS, who speak all three languages. We thus have a collection of units (pairs or individuals) each containing one speaker of each language.
If ES < F, then after the pairing off we are left with equal numbers of members of E, F, and S. These may be formed into triplets, with each triplet containing one speaker of each language. As before we also have the students in EFS. Again, we have partitioned the student body into units with each unit containing one speaker of each language.
If ES > F, then after the pairing off, we are left with an equal number of members of ES, FS and EF. These may be formed into triplets, with each triplet containing two speakers of each language. So, in this case we partition the student body into units with each unit containing either one speaker of each language, or two speakers of each language.
Finally, we may divide the units into 5 groups with 10 speakers of each language in each group.

Problem 98
Hint $2/(x^2 - 1) = 1/(x - 1) - 1/(x + 1)$ etc.
Solution lhs = $1/(x - 1) - 1/(x + 1) + 1/(x - 2) - 1/(x + 2) + ... + 1/(x + 10) - 1/(x - 10) = 1/(x - 1) - 1/(x + 10) + 1/(x - 2) - 1/(x + 9) + ... + 1/(x - 10) - 1/(x + 1)$ = rhs.

Problem 99
Hint Obtain formulae for the two lengths.
Solution Answer: n = 9.

For n < 6, there is at most one length of diagonal. For n = 6, 7 the longest and shortest, and a side of the n-gon form a triangle, so the difference between the longest and shortest is less than the side.
For n > 7 the side has length 2R sin pi/n, the shortest diagonal has length 2R sin 2pi/n, and the longest diagonal has length 2R for n even and 2R cos pi/2n for n odd (where R is the radius of the circumcircle). Thus we require:
sin 2pi/n + sin pi/n = 1 and n even, or
sin 2pi/n + sin pi/n = cos pi/2n and n odd.
Evidently the lhs is a strictly decreasing function of n and the rhs is an increasing function of n, so there can be at most one solution of each equation. The second equation is satisfied by n = 9, although it is easier to see that there is a quadrilateral with the longest diagonal and shortest diagonals as one pair of opposite sides, and 9-gon sides as the other pair of opposite sides. The angle between the longest side and an adjacent side is 60, so that its length is the length of the shortest diagonal plus 2 x 9-gon side x cos 60. Hence that is the only solution for n odd.
For n = 8 we have the same quadrilateral as for the 9-gon except that the angle is 67.5 and hence the difference is less than 1. For n = 10, sin 2pi/10 + sin pi/10 = sin pi/10 (2 cos pi/10 + 1) < 3 sin pi/10 < 3 pi/10 < 1. So there are no solutions for n even >= 10, and hence no solutions for n even.

Problem 100
Hint Square.
Solution First we must notice that for 1 <= a, b we have a < b, then a + 1/a < b + 1/b.
This is basic to any estimation.
The obvious approach is to notice that if $a_i \le n$, then $a_{i+1} \ge a_i + 1/n$. Hence it takes at most n steps to get from n - 1 to n. Unfortunately, this does not quite work: we need 2 + 3 + ... + 14 = 104 steps to get from 1 to 14.
The trick is to notice that $a_{n+1}^2 > a_n^2 + 2$. But $a_2 = 2$, so $a_n^2 > 2n$. That gives $a_{100}^2 > 200 > 14^2$.

Problem 102
Hint Divide the number by n.
Solution Given m <= n! write m = nq + r (*), with 0 <= q, 0 <= r < m. Then q <= (n-1)!, so q is a sum of at most n-1 distinct factors of (n-1)!. r is itself a factor of n! and is not divisible by n, so (*) expresses m as a sum of at most n distinct factors of n!.

Problem 103
Hint We need to show that angle A = 2pi/5.
Solution DA = DE, so DAE is isosceles. DE is parallel to BC, so ABC is isosceles, so BA = AC/(2 cos A). Hence BD = AC/(2 cos A) - AC. But AE = 2 AC cos A, so we have an equation for c=cos A: $4c^2 + 2c - 1 = 0$.
2pi/5, 4pi/5, 6pi/5, 8pi/5 and 10pi/5 are the roots of: real part of $(\cos\theta + i \sin\theta)^5 = 1$.
Expanding this gives that cos 2pi/5, cos 4pi/5, cos 6pi/5, cos 8pi/5 and 1 are the roots of $16c^5 - 20c^3 + 5c - 1 = 0$. Dividing by (c - 1) gives $16c^4 + 16c^3 - 4c^2 - 4c + 1 = (4c^2 + 2c - 1)^2$. So cos 2pi/5 (= cos 8pi/5) and cos 4pi/5 (= cos 6pi/5) are the roots of $4c^2 + 2c - 1 = 0$.
We know that A < 90 degrees (since A = C and their sum is less than 180 degrees). Hence A = 2pi/5. So BD = 2 AC cos 2pi/5 = 2 AC sin pi/10, which is the side length for a regular 10-gon inscribed in a circle radius AC.

Problem 104
Hint The center belongs to all three spheres. Use convexity.

Solution Let the tetrahedron have side 1. Then the center O is a distance $1/\sqrt{8}$ from the center of each of the spheres, so it is contained in each of the spheres. We now use convexity.
Two circles with diameters two of the sides of a triangle cover the triangle (consider the foot of the altitude to the third side), so faces ABC and ABD are certainly contained in the spheres. Consider face ACD. The sphere on BC passes through the midpoints of AC and CD, and through C, so it contains the triangle formed by these three points (by convexity). But the rest of ACD is contained in the sphere on AD. Similarly for the face BCD. Hence all the faces are contained in the spheres. But now take any point P inside the tetrahedron. Extend OP to meet a face at X. X lies in one of the spheres, but O also lies in the sphere and hence all points on OX, including P (by convexity).
False in general. Take ABCD to be a plane square, then no points on CD are in the spheres except C and D (and we can obviously distort this slightly to make it less degenerate).

Problem 105
Hint Consider the set of positions on the edges but not at the corners.
Solution

(a) Let S be the set of the 8 positions on the perimeter not at a corner. Any move changes the sign of either 2 or 0 of the members of S. We start with an odd number of members of S with a minus sign, so we must always have an odd number of members of S with a minus sign and hence cannot get all plus signs.
(b) Also impossible. The same argument works.

Problem 106
Hint The small triangles have equal area. Remember that if two medians are equal, then the triangle is isosceles.
Solution Denote the side lengths by a, b, c and the corresponding median lengths by m_a, m_b, m_c. The six small triangles all have equal area. [Let the areas be $t_1, \dots, t_6$. It is obvious that the adjacent pairs have equal height and equal base, so we have $t_1 = t_2$, $t_3 = t_4$, $t_5 = t_6$. The three on each side of a median sum to the same area, so $t_1 + t_2 + t_3 = t_4 + t_5 + t_6$, $t_1 + t_5 + t_6 = t_2 + t_3 + t_4$. Subtracting gives $t_1 = t_4$. Similarly, $t_2 = t_5$ and we are home.] So by the usual result that the twice area of a triangle equals its perimeter times its in-radius, we conclude that the perimeters of four of the small triangles are equal.
Two of them must share a side of the original triangle. Suppose it is a. Then we have: $a/2 + m_a/3 + 2m_b/3 = a/2 + m_a/3 + 2m_c/3$. So $m_b = m_c$. That implies that b = c. [Because the triangle formed by the centroid and side a is isosceles, so the median is perpendicular to the side, so the main triangle is isosceles.]
Using the facts that b = c and $m_b = m_c$, we see that two of the remaining small triangles have perimeter $b/2 + m_b$ and two have perimeter $b/2 + m_b/3 + 2m_a/3$. So there are two cases to consider. In the first case $a/2 - m_a/3 = b/2 - m_b/3$. That implies a = b, since if a < b, $m_a > m_b$ (consider the triangle formed by the centroid and the side c). So the triangle is equilateral.
The second case is harder. We have: $a/2 + m_a/3 + 2m_b/3 = b/2 + m_b$, and hence $a/2 + m_a/3 = b/2 + m_b/3$ (*). Take the angle between a and b to be θ. Then $m_a = b \sin\theta$, $a = 2b\cos\theta$, and $m_b^2 = b^2/4 + a^2 - ab\cos\theta = b^2/4 + 2b^2\cos^2\theta$. We can now use (*) to get an equation for θ. First we square (*) to get: $m_b^2 = (3a/2 - 3b/2 + m_a)^2$. We divide out the factor b^2 to get: $1/4 + 2\cos^2\theta = 3 1/4 + 8\cos^2\theta - 9\cos\theta + 3\sin\theta(2\cos\theta - 1)$. Squaring, so that we can use $\sin^2\theta = 1 - \cos^2\theta$, and writing $c = \cos\theta$, we get: $(1 - c^2)(4c^2 - 4c + 1) = 4c^4 - 12c^3 + 13c^2 - 6c + 1$. Hence $8c^4 - 16c^3 + 10c^2 - 2c = 0$. Factorizing: $c(c - 1)(2c - 1)^2 = 0$. c = 0 and c = 1 give degenerate triangles, so we must have c = 1/2 and hence the triangle is equilateral.

Problem 107
Hint $x^2 + x + 1 = x(x + 1) + 1$ = multiple of x + 1.
Solution This is a trivial variant on the proof that there are an infinite number of primes. Suppose that we can only find x, y for a finite number of primes $p_1, p_2, \dots, p_n$. Set $x = p_1p_2\dots p_n$. Then none of the p_i can divide $x(x+1) + 1$. But it must have prime factors. Contradiction.

Problem 108
Hint Answer: 24.
Consider the competitors who get one or more rank 1s.
Solution Answer: 24.
At most 4 competitors can receive a rank 1. For a competitor with a rank 1 can only receive ranks 1, 2, 3 or 4. There are only 36 such ranks available and each competitor with a rank 1 needs 9 of them.
If only one competitor receives a rank 1, then his score is 9. If only 2 competitors receive a rank 1, then one of them must receive at least five rank 1s. His maximum score is then 5.1 + 4.4 = 21. If 4 competitors receive a rank 1, then they must use all the 36 ranks 1, 2, 3, and 4. The total score available is thus 9(1 + 2 + 3 + 4) = 90, so at least one competitor must receive 22 or less. Thus the winner's maximum score is at most 22. If 3 competitors receive a rank 1, then the winner's score is maximised by giving all three competitors the same score and letting them share the 27 ranks 1, 3 and 4. That gives a winner's score of 9(1 + 3 + 4)/3 = 24. That can be achieved in several ways, for example: each competitor gets 3 1s, 3 3s and 3 4s, or one competitor gets 4 1s and 5 4s, another gets 3 1s, 3 3s and 3 4s, another gets 2 1s 6 3s and one 4. Note that it is trivial to arrange ranks for the remaining 17 competitors. For example: give one 5 2s and 4 5s total 30, one 4 2s and 5 5s total 33, and then one 9 6s, one 9 7s and so on.
Thus the answer is 24, with three joint winners. If there is required to be a single winner, then the answer is 23.

Problem 113
Hint The cosine rule gives us that most angles have rational cosine, but that is not enough. Find a relation involving OA, OB and rationals.
Solution AB = AO cos OAB + BO cos OBA. We can derive a rational expression for cos OAB using the cosine rule for triangle ABC. Similarly for cos OBA using the cosine rule for triangle DAB. So $OA = r_1 + r_2OB$, where r_i denotes a rational number. Similarly, $OB = r_3 + r_4OC$, so $OA = r_5 + r_6OC$. But $OA + OC = AC = r_7$. Hence OA is rational.