## 1959

**Problem 1**
**Solution** 3(14n+3)-2(21n+4)=1.

**Problem 2**
**Solution** Care is needed to take the positive root. Taking one term across to the other side before squaring is a bad idea, because the product of the two main roots simplifies to |x-1|.
We find: (a) any x in the interval [1/2,1]; (b) no solutions (3/4 is not a solution); (c) x=3/2.

**Problem 3**
**Solution** You need that cos 2x = 2 $\cos^2$x - 1. Some easy manipulation then gives:
$a^2\cos^2 2x + (2a^2 + 4ac - 2b^2) \cos 2x + (4c^2 + 4ac - 2b^2 + a^2) = 0$.
The equations are the same for the values of a, b, c given. The angles are 2pi/5 (or 8pi/5) and 4pi/5 (or 6pi/5).

**Problem 4**
**Solution** Area = AB.BC/2 = $BM^2/2 = AC^2/8$ (because BM = AM = MC), so B lies a distance AC/4 from AC. Take B as the intersection of a circle diameter AC with a line parallel to AC distance AC/4.

**Problem 5**
**Solution**
(a) Angle ANM = angle BNM = 45 (half the angle at the center of the respective circle). So N lies on the semicircle diameter AB. We can also easily check that angle ANM + angle FNM = 180, so A, N, F are collinear. A similar idea works for B, C, N.
(b) Let NM meet the circle diameter AB again at X. Angle ANS = angle BNS implies AS = BS and hence S is a fixed point.
(c) Clearly the height of the midpoint above AB is AB/4. Since it varies continuously with M, it must be the interval between the two extreme positions, so the locus is a segment length AB/2 centered over AB.

**Problem 6**
**Solution** Let the planes meet in the line L. Then AB and CD must be parallel to L. Let H be the foot of the perpendicular from C to AB. The fact that a circle can be inscribed implies AH = AD = BC. This gives enough to construct ABCD. Note that if CH>AH then no construction is possible. If CH<AH, then there are two solutions, one with AB>CD, the other with AB<CD.

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## 1960

**Problem 1**
**Solution** Answer: 550, 803.
Personally, I hate this type of question. The fastest way to solve it is almost certainly to scan the 81 multiples of 11 from 110 to 990. However, to get much credit you have to adopt a more algebraic approach. But a question which allows the enumeration approach is a bad question.





So, put N/11 = 10a + b. If a + b <= 9, we have $2a^2 + 2ab + 2b^2 = 10a + b$, so b is odd and after a little more work we find 550 is the only solution. If a + b > 9, we have $(a+1)^2 + (a+b-10)^2 + b^2 = 10a + b$, so b is odd and after a little more work we find 803 is the only solution.

**Problem 2**
**Solution** Answer: - 1/2 <= x < 45/8.
We require the first inequality to avoid imaginary numbers. Hence we may set x = $-1/2 + a^2/2$, where a >= 0. The inequality now gives immediately a < 7/2 and hence x < 45/8. It is a matter of taste whether to avoid x=0. I would allow it because the limit as x tends to 0 of the lhs is 4, and the inequality holds.

**Problem 3**
**Solution** Let M be the midpoint of BC, and P and Q the two points a/2n either side of it, with P nearer B. Then $\alpha$ = angle PAQ = angle QAH - angle PAH (taking angles as negative if P (or Q) lies to the left of H). So $\tan\alpha = (QH - PH)/(AH^2 + QH.PH) = AH.PQ/(AH^2 + (MH - a/2n)(MH + a/2n)) = (ah/n)/(a^2/4 - a^2/(4n^2)) = 4nh/(an^2 - a)$.

**Problem 4**
**Solution** Let M be the midpoint of BC, AH the altitude from A, and BI the altitude from B. Start by constructing AHM. Take X on the circle diameter AM with MX = BI/2. Let the lines AX, HM meet at C and take B so that BM = MC. [This works because CMX and CBI are similar with MX = BI/2 and hence CM = CB/2.]

**Problem 5**
**Solution** The key idea is that the midpoint must lie in the plane half-way between ABCD and A'B'C'D'. Similarly, Z must lie in the plane one-third of the way from ABCD to A'B'C'D'.
(a) Regard ABCD as horizontal. Then the locus is the square with vertices the midpoints of the vertical sides.
(b) The locus is a rectangle $\sqrt{2}/3 \times 2\sqrt{2}/3$.

**Problem 6**
**Solution** Let the radius of the sphere be r and the half-angle of the cone $\theta$. Then the the cone's height is $(1 + 1/\sin\theta)$, and the radius of its base is $r(1 + 1/\sin\theta)\tan\theta$. We easily find $V_1/V_2 = (1 + s)^2/(s - s^2)$, where s = sin $\theta$. So $V_1/V_2 = 1$ requires $2s^2 + s + 1 = 0$, which is impossible since 0 <= s < 1. Differentiating, we find the minimum is at s = 1/3.

**Problem 7**
**Solution** Since angle BXC = $90^o$, X lies on the circle diameter BC. In general this will intersect the axis of symmetry in 0, 1 or 2 points. By symmetry any points of intersection X will also lie on the circle diameter AD and so will have angle AXD = $90^o$ also.
Let L be the midpoint of AB, and M the midpoint of CD. Let X lie on LM a distance x from L. We have LB = a/2, MC = c/2, and XM = h - x. The triangles LBX and MXC are similar, so 2x/a = c/(2(h-x)). Hence $4x^2 - 4xh + ac = 0$, so $x = h/2 \pm (\sqrt{(h^2 - ac)})/2$.
There are 0, 1, 2 points according as $h^2$ <, =, > ac.

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## 1961

**Problem 1**
**Solution** A routine slog gives $z = (a^2 - b^2)/2a$, x and $y = (a^2 + b^2)/4a \pm \sqrt{(10a^2b^2 - 3a^4 - 3b^4)}/4a$.
A little care is needed with the conditions. Clearly x, y, z positive implies $a > 0$, and then z positive implies $|b| < a$. The expression under the root must be positive. It helps if you notice that it factorizes as $(3a^2 - b^2)(3b^2 - a^2)$. The second factor is positive because $|b| < a$, so the first factor must also be positive and hence $a < \sqrt{3}.|b|$. These conditions are also sufficient to ensure that x and y are distinct, but then z must also be distinct because $z^2 = xy$.

**Problem 2**
**Solution** One approach is a routine slog from Heron's formula. The inequality is quickly shown to be equivalent to $a^2b^2 + b^2c^2 + c^2a^2 <= a^4 + b^4 + c^4$, which is true since $a^2b^2 <= (a^4 + b^4)/2$. We get equality iff the triangle is equilateral.
Another approach is to take an altitude lying inside the triangle. If it has length h and divides the base into lengths r and s, then we quickly find that the inequality is equivalent to $(h - (r + s)\sqrt{3}/2)^2 + (r - s)^2 >= 0$, which is true. We have equality iff r = s and $h = (r + s)\sqrt{3}/2$, which means the triangle is equilateral.
*A third solution:*
We have $b^2 + c^2 >= 2bc$ with equality iff b = c. Also for any angle x in the range $0^o$ to $180^o$ we have $2bc >= 2bc \sin(X + 30^o)$ with equality iff $X = 60^o$. So taking X to be the angle between the sides b and c (we cannot call it A because A is already used to mean the area in this question!) we have that $b^2 + c^2 >= bc \sin(X + 30^o)$ with equality iff the triangle is equilateral. Now $2 \sin(X + 30^o) = \sqrt{3} \sin X + \cos X$, so using the cosine rule $a^2 = b^2 + c^2 - 2bc \cos X$, we get the required inequality.

**Problem 3**
**Solution** Since $\cos^2x + \sin^2x = 1$, we cannot have solutions with n not 2 and $0 < |\cos x|, |\sin x| < 1$. Nor can we have solutions with n=2, because the sign is wrong. So the only solutions have sin x = 0 or cos x = 0, and these are: x = multiple of pi, and n even; x even multiple of pi and n odd; x = even multiple of pi + 3pi/2 and n odd.

**Problem 4**
**Solution** Take lines through the centroid parallel to the sides of the triangle. The result is then obvious.

**Problem 5**
**Solution** The key is to take N so that A is the midpoint of NB, then angle NCB = $\alpha$.
The construction is as follows: take BN length 2AB. Take circle through B and N such that the angle BPN = $\alpha$ for points P on the arc BN. Take A as the midpoint of BN and let the circle center A, radius AC cut the arc BN at C. In general there are two possibilities for C.
Let X be the intersection of the arc BN and the perpendicular to the segment BN through A. For the construction to be possible we require AX >= AC > AB. But AB/AX = $\tan \alpha/2$, so we get the condition in the question.
**Problem 6**
**Solution** The key is to notice that O is the midpoint of the segment joining the centroids





of ABC and A'B'C'. The centroid of ABC is fixed, so the locus is just the plane parallel to p and midway between p and the centroid of ABC.

## 1962

**Problem 1**
**Solution** We have $4(10n+6) = 6.10^m + n$, where n has m digits. So $13n + 8 = 2.10^m$. Hence n = 2n' and $13n' = 10^m - 4$. Dividing, we quickly find that the smallest n', m satisfying this are: n'=7692, m=5. Hence the answer is 153846.

**Problem 2**
**Solution** It is easy to show that the inequality implies $|x-1| > \sqrt{31}/8$, so $x > 1 + \sqrt{31}/8$, or $x < 1 - \sqrt{31}/8$. But the converse is not true.
Indeed, we easily see that x > 1 implies the lhs < 0. Also care is needed to ensure that the expressions under the root signs are not negative, which implies $-1 <= x <= 3$. Putting this together, suggests the solution is $-1 <= x < 1 - \sqrt{31}/8$, which we can easily check.

**Problem 3**
**Solution** The rhombus WXCZ, where W is the midpoint of AB', X is the midpoint of BC', and Z is the midpoint of BD'.

**Problem 4**
**Solution** Put c = cos x, and use $\cos 3x = 4c^3 - 3c$, $\cos 2x = 2 c^2 - 1$. We find the equation given is equivalent to c = 0, $c^2 = 1/2$ or $c^2 = 3/4$. Hence x = pi/2, 3pi/2, pi/4, 3pi/4, pi/6, 5pi/6 or any multiple of pi plus one of these.

**Problem 5**
**Solution** The key is to notice that if O is the center of the inscribed circle, then angle AOC = 270 - angle ABC (chase a few angles around and use the fact that opposite angles in a cyclic quadrilateral sum to 180). So O must be the intersection of the arc AC and the angle bisector of angle ABC. To prove the construction possible we use the fact that a quadrilateral ABCD has an inscribed circle iff AB + CD = BC + AD. For D near C on the circumcircle of ABC we have AB + CD < BC + AD, whilst for D near A we have AB + CD > BC + AD, so as D moves continuously along the circumcircle there must be a point with equality. [Proof that the condition is sufficient: it is clearly necessary (use fact that tangents from a point are of equal length). So take a circle touching AB, BC and AD and let the other tangent from C (not BC) meet AD in D'. Then CD' - CD = AD' - AD, hence D'=D.]

**Problem 6**
**Solution** Let the triangle be ABC with AB = AC, let the incenter be I and the circumcenter O. Let the distance IO be d, taking d positive if O is closer to A than I, negative if I is closer. Let the angle OAB be $\theta$.
Then $r = (R + d) \sin \theta$, and $r + d = R \cos 2\theta$. It helps to draw a figure to check that this remains true for the various possible configurations. Using $\cos 2\theta = 1 - 2 \sin^2\theta$, we find that $(d + R + r)(d^2 - R(R - 2r)) = 0$. But OI < OA, so d is not - R - r. Hence result.

**Problem 7**
**Solution** First part is obvious. The wrong way to do the second part is to start looking for the locus of the center of a sphere which touches three edges. The key is to notice that the tangents to a sphere from a given point have the same length.
Let the tetrahedron be $A_1A_2A_3A_4$. Let S be the sphere inside the tetrahedron, $S_1$ the tetrahedron opposite $A_1$, and so on. Let the tangents to S from $A_i$ have length $a_i$. Then the side $A_iA_j$ has length $a_i+a_j$. Now consider the tangents to $S_1$ from $A_1$. Their lengths are $a_1 + 2a_2$, $a_1 + 2a_3$, and $a_1 + 2a_4$. Hence $a_2 = a_3 = a_4$. Similarly, considering $S_2$, we have that $a_1 = a_3 = a_4$.

## 1963

**Problem 1**
**Solution** I must admit to having formed rather a dislike for this type of question which came up in almost every one of the early IMOs. Its sole purpose seems to be to teach you to be careful with one-way implications: the fact that $a^2 = b^2$ does not imply a = b.
The lhs is non-negative, so x must be non-negative. Moreover $2\sqrt{(x^2 - 1)} <= x$, so $x <= 2/\sqrt{3}$. Also $\sqrt{(x^2 - p)} <= x$, so $p >= 0$.
Squaring etc gives that any solution must satisfy $x^2 = (p - 4)^2/(16 - 8p)$. We require $x <=2/\sqrt{3}$ and hence $(3p - 4)(p + 4) <= 0$, so $p <= 4/3$.
Substituting back in the original equality we get $|3p-4| + 2|p| = |p - 4|$, which is indeed true for any p satisfying $0 <= p <= 4/3$.

**Problem 2**
**Solution** Take the solid sphere on diameter AB, and the solid sphere on diameter AC. Then the locus is the points in one sphere but not the other (or on the surface of either sphere). Given P, consider the plane through P perpendicular to AP and the parallel planes through the other two points of intersection of AP with the two spheres (apart from A) which pass through B and C.

**Problem 3**
**Solution** For n odd consider the perpendicular distance of the shortest side from the opposite vertex. This is a sum of terms $a_i$ x cosine of some angle. We can go either way round. The angles are the same in both cases, so the inequalities give that $a_1 = a_{n-1}$, and hence $a_1 = a_i$ for all i < n. We get $a_1 = a_n$ by repeating the argument for the next shortest side. The case n even is easier, because we take a line through the vertex with sides $a_1$ and $a_n$ making equal angles with them and look at the perpendicular distance to the opposite vertex. This gives immediately that $a_1 = a_n$.

**Problem 4**
**Solution** Successively eliminate variables to get $x_1(y - 2)(y^2 + y - 1)^2 = 0$. We have the trivial solution $x_i = 0$ for any y. For y = 2, we find $x_i = s$ for all i (where s is arbitary). Care is needed for the case $y^2 + y - 1 = 0$, because after eliminating three variables the two remaining equations have a factor $y^2 + y - 1$, and so they are automatically satisfied. In this case, we can take any two $x_i$ arbitary and still get a solution. For example, $x_1 = s$, $x_2 = t$, $x_3 = - s + yt$, $x_4 = - ys - yt$, $x_5 = ys - t$.

**Problem 5**
**Solution** Consider the roots of $x^7 + 1 = 0$. They are $e^{ipi/7}$, $e^{i3pi/7}$, ... , $e^{i13pi/7}$ and must have





sum zero since there is no $x^6$ term. Hence, in particular, their real parts sum to zero. But cos7pi/7 = - 1 and the others are equal in pairs, because cos(2pi - x) = cos x. So we get cos pi/7 + cos 3pi/7 + cos 5pi/7 = 1/2. Finally since cos(pi - x) = - cos x, cos 5pi/7 = - cos 2pi/7.

**Problem 6**
**Solution** I cannot see an elegant solution. Start from the second prediction. The disjoint pairs can only be: DA, EC; DC, CB; or AE, CB. The additional requirement of just two correct places means that the only possibilities (in the light of the information about the second predicition) are: DABEC, DACBE, EDACB, AEDCB. The first is ruled out because AB are consecutive. The second is ruled out because C is in the correct place. The fourth is ruled out because A is in the correct place. This leaves EDACB, which is indeed a solution.

## 1964

**Problem 1**
**Solution** $2^3 = 1 \pmod 7$. Hence $2^{3m} = 1 \pmod 7$, $2^{3m+1} = 2 \pmod 7$, and $2^{3m+2} = 4 \pmod 7$. Hence we never have 7 dividing $2^n + 1$, and 7 divides $2^n - 1$ iff 3 divides n.

**Problem 2**
**Solution** The condition that a, b, c be the sides of a triangle, together with the appearance of quantities like a + b - c is misleading. The inequality holds for any a , b, c >= 0.
At most one of (b+c-a), (c+a-b), (a+b-c) can be negative. If one of them is negative, then certainly:
abc >= (b + c - a)(c + a - b)(a + b - c) (*)
since the lhs is non-negative and the rhs is non-positive.
(*) is also true if none of them is negative. For then the arithmetic/geometric mean on b + c - a, c + a - b gives:
$c^2 >= (b + c - a)(c + a - b)$.
Similarly for $a^2$ and $b^2$. Multiplying and taking the square root gives (*). Multiplying out easily gives the required result.

**Problem 3**
**Solution** This is easy once you realize that the answer is not nice and the derivation a slog. Use r = 2.area/perimeter and Heron's formula: area k is given by $16k^2 = (a + b + c)(b + c - a)(c + a - b)(a + b - c)$. The small triangles at the vertices are similar to the main triangle and smaller by a factor (h - 2r)/h, where h is the relevant altitude. For the triangle opposite side a:
(h - 2r)/h = 1 - 2(2k/p)/(2k/a) = 1 - 2a/p = (b + c - a)/(a + b + c). Hence the total area is $((a + b + c)^2 + (b + c - a)^2 + (c + a - b)^2 + (a + b - c)^2)/(a + b + c)^2$ pi $r^2 = (a^2 + b^2 + c^2)$.pi.$(b + c - a)(c + a - b)(a + b - c)/(a + b + c)^3$.
**Problem 4**
**Solution** Take any person. He writes to 16 people, so he must write to at least 6 people on the same topic. If any of the 6 write to each other on that topic, then we have a group of three writing to each other on the same topic. So assume they all write to each other on the other two topics. Take any of them, B. He must write to at least 3 of the other 5 on the same topic. If two of these write to each other on this topic, then they form a

group of three with B. Otherwise, they must all write to each other on the third topic and so from a group of three.

**Problem 5**
**Solution** It is not hard to see that the required number is at most 315. But it is not at all obvious how you prove it actually is 315, short of calculating the 315 points intersection for a specific example.
Call the points A, B, C, D, E. Given one of the points, the other 4 points determine 6 lines, so there are 6 perpendiculars through the given point and hence 30 perpendiculars in all. These determine at most 30.29/2 = 435 points of intersection. But some of these necessarily coincide. There are three groups of coincidences. The first is that the 6 perpendiculars through A meet in one point (namely A), not the expected 15. So we lose 5.14 = 70 points. Second, the lines through C, D and E perpendicular to AB are all parallel, and do not give the expected 3 points of intersection, so we lose another 10.3 = 30 points. Third, the line through A perpendicular to BC is an altitude of the triangle ABC, as are the lines through B perpendicular to AC, and the through C perpendicular to AB. So we only get one point of intersection instead of three, thus losing another 10.2 = 20 points. These coincidences are clearly all distinct (the categories do not overlap), so they bring us down to a maximum of 435 - 120 =315.
There is no obvious reason why there should be any further coincidences. But that is not quite the same as proving that there are no more. Indeed, for particular positions of the points A, B, C, D, E we can certainly arrange for additional coincidences (the constraints given in the problem are not sufficient to prevent additional coincidences). So we have to prove that it is possible to arrange the points so that there are no additional coincidences. I cannot see how to do this, short of exhibiting a particular set of points, which would be extremely tiresome.

**Problem 6**
**Solution** Yes, indeed it is true for an arbitary point in the plane of ABC not on any of the lines AB, BC, CA
Take D as the origin. Let A, B, C be the points **a**, **b**, **c** respectively. Then $D_0$ is p**a** + q**b** + r**c** with p + q + r = 1 and p, q, r > 0. So a point on the line parallel to $DD_0$ through A is **a** + s(p**a** + q**b** + r**c**. It is also in the plane DBC if s = -1/p, so $A_0$ is the point - q/p **b** - r/p **c**. Similarly, $B_0$ is - p/q **a** - r/q **c**, and $C_0$ is - p/r **a** - q/r **b**.
The volume of ABCD is 1/6 |**a x b.c**| and the volume of $A_0B_0C_0D_0$ is 1/6 |(p**a** + (q + q/p)**b** + (r + r/p)**c**) **x** ((p + p/q)**a** + q**b** + (r + r/q)**c**).((p + p/r)**a** + (q + q/r)**b** + r**c**)|
Thus vol $A_0B_0C_0D_0$/vol ABCD = abs value of the determinant:
```
| p        q + q/p  r + r/p |
| p + p/q  q        r + r/q |
| p + p/r  q + q/r  r       |
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which is easily found to be 2 + p + q + r = 3.

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## 1965

**Problem 1**
**Solution** Let y = |√(1 + sin 2x) - √(1 - sin 2x)|. Then $y^2$ = 2 - 2|cos 2x|. If cos x is positive, then since 2 cos x <= ... <= √2, 2 $\cos^2$x <= 1 and hence |cos 2x| = |2 $\cos^2$x - 1| = 1 - 2 $\cos^2$x, so y = 2 cos x. In this case the inequalities reduce to cos x < = 1/√2, and hence x lies in the interval [pi/4, pi/2] or in the interval [3pi/2, 7pi/4].





If cos x < 0, then we cannot deduce that |cos x| <= 1/√2. If it is, then we get cos x <= |cos x| <= 1/√2 and x lies in the interval [pi/2, 3pi/4] or in the interval [5pi/4, 3pi/2]. If it is not, then |cos 2x| = 2 $\cos^2$x - 1, and so y = 2|sin x| and hence x lies in the interval [3pi/4, 5pi/4].
Thus the complete solution is that x lies in the interval [pi/4, 7pi/4].

**Problem 2**
**Solution** The slog solution is to multiply out the determinant and show it is non-zero. A slicker solution is to take the $x_i$ with the largest absolute value. Say $|x_1|$ >= $|x_2|$, $|x_3|$. Then looking at the first equation we have an immediate contradiction, since the first term has larger absolute value than the sum of the absolute values of the second two terms.

**Problem 3**
**Solution** Let the plane meet AD at X, BD at Y, BC at Z and AC at W. Take plane parallel to BCD through WX and let it meet AB in P. Since the distance of AB from WXYZ is k times the distance of CD, we have that AX = k.XD and hence that AX/AD = k/(k+1). Similarly AP/AB = AW/AC = AX/AD. XY is parallel to AB, so also AX/AD = BY/BD = BZ/BC.
vol ABWXYZ = vol APWX + vol WXPBYZ. APWX is similar to the tetrahedron ABCD. The sides are k/(k+1) times smaller, so vol APWX = $k^3(k+1)^3$ vol ABCD. The base of the prism WXPBYZ is BYZ which is similar to BCD with sides k/(k+1) times smaller and hence area $k^2(k+1)^2$ times smaller. Its height is 1/(k+1) times the height of A above ABCD, so vol prism = 3 $k^2(k+1)^3$ vol ABCD. Thus vol ABWXYZ = ($k^3$ + $3k^2$)/$(k+1)^3$ vol ABCD. We get the vol of the other piece as vol ABCD - vol ABWXYZ and hence the ratio is (after a little manipulation) $k^2$(k+3)/(3k+1).

**Problem 4**
**Solution** Answer: 1,1,1,1 or 3,-1,-1,-1.
Let the numbers be $x_1$, ... , $x_4$. Let t = $x_1x_2x_3x_4$. Then $x_1$ + t/$x_1$ = 2. So all the $x_i$ are roots of the quadratic $x^2$ - 2x + t = 0. This has two roots, whose product is t.
If all $x_i$ are equal to x, then $x^3$ + x = 2, and we must have x = 1. If not, then if $x_1$ and $x_2$ are unequal roots, we have $x_1x_2$ = t and $x_1x_2x_3x_4$ = t, so $x_3x_4$ = 1. But $x_3$ and $x_4$ are still roots of $x^2$ - 2x + t = 0. They cannot be unequal, otherwise $x_3x_4$ = t, which gives t = 1 and hence all $x_i$ = 1. Hence they are equal, and hence both 1 or both -1. Both 1 gives t = 1 and all $x_i$ = 1. Both -1 gives t = -3 and hence $x_i$ = 3, -1, -1, -1 (in some order).

**Problem 5**
**Solution** Let X be the foot of the perpendicular from B to OA, and Y the foot of the perpendicular from A to OB. We show that the orthocenter of OPQ lies on XY.
MP is parallel to BX, so AM/MB = AP/PX. Let H be the intersection of XY and the perpendicular from P to OB. PH is parallel to AY, so AP/PX = YH/HX. MQ is parallel to AY, so AM/MB = YQ/BQ. Hence YQ/BQ = YH/HX and so QH is parallel to BX and hence perpendicular to AO, so H is the orthocenter of OPQ as claimed.
If we restrict M to lie on a line A'B' parallel to AB (with A' on OA, B' on OB) then the locus is a line X'Y' parallel to XY, so as M moves over the whole interior, the locus is the interior of the triangle OXY.

**Problem 6**

**Solution** The key is that if two segments length d do not intersect then we can find an endpoint of one which is a distance > d from an endpoint of the other.
Given this, the result follows easily by induction. If false for n, then there is a point A in three pairs AB, AC and AD of length d (the maximum distance). Take AC to lie between AB and AD. Now C cannot be in another pair.
Suppose it was in CX. Then CX would have to cut both AB and AD, which is impossible. To prove the result about the segments, suppose they are PQ and RS. We must have angle PQR less than 90, otherwise PR > PQ = d. Similarly, the other angles of the quadrilateral must all be less than 90. Contradiction.

## 1966

**Problem 1**
**Solution** Answer: 6. Let a solve just A, b solve just B, c solve just C, and d solve B and C but not A. Then 25 - a - b - c - d solve A and at least one of B or C. The conditions give:
b + d = 2(c + d); a = 1 + 25 - a - b - c - d; a = b + c.
Eliminating a and d, we get: 4b + c = 26. But d = b - 2c >= 0, so b = 6, c = 2.

**Problem 2**
**Solution** A straight slog works. Multiply up to get (a + b) cos A cos B cos C/2 = a sin A cos B sin C/2 + b cos A sin B sin C/2 (where a = BC, b = AC, as usual). Now use cos(A + C/2) = cos A cos C/2 - sin A sin C/2 and similar relation for cos (B + C/2) to get: a cos B cos(A + C/2) + b cos A cos (B + C/2) = 0. Using C/2 = 90 - A/2 - B/2, we find that cos(A + C/2) = - cos(B + C/2) (and = 0 only if A = B). Result follows.

**Problem 3**
**Solution** Let the tetrahedron be ABCD and let P be a general point. Let X be the midpoint of CD. Let P' be the foot of the perpendicular from P to the plane ABX. We show that if P does not coincide with P', then PA + PB + PC + PD > P'A + P'B + P'C + P'D.
PA > P'A (because angle PP'A = 90) and PB > P'B. P'CD is isosceles and PCD is not but P is the same perpendicular distance from the line CD as P'. It follows that PC + PD > P'C + P'D. The easiest way to see this is to reflect C and D in the line PP' to give C' and D'. Then PC = PC', and PC' + PD > C'D = P'C' + P'D = P'C + P'D.
So if P has the smallest sum, it must lie in the plane ABX and similarly in the plane CDY, where Y is the midpoint of AB, and hence on the line XY. Similarly, it must lie on the line joining the midpoints of another pair of opposite sides and hence must be the center.

**Problem 4**
**Solution** cot y - cot 2y = cos y/sin y - (2 $\cos^2$y - 1)/(2 sin y cos y) = 1/(2 sin y cos y) = 1/sin 2y. The result is now easy. Use induction. True for n = 1 (just take y = x). Suppose true for n, then taking y = $2^n$x, we have 1/sin $2^{n+1}$x = cot $2^n$x - cot $2^{n+1}$x and result follows for n + 1.

**Problem 5**
**Solution** Take $a_1 > a_2 > a_3 > a_4$. Subtracting the equation for i=2 from that for i=1 and dividing by ($a_1 - a_2$) we get:





$- x_1 + x_2 + x_3 + x_4 = 0$.
Subtracting the equation for i=4 from that for i=3 and dividing by ($a_3 - a_4$) we get:
$- x_1 - x_2 - x_3 + x_4 = 0$.
Hence $x_1 = x_4$. Subtracting the equation for i=3 from that for i=2 and dividing by ($a_2 - a_3$) we get:
$- x_1 - x_2 + x_3 + x_4 = 0$.
Hence $x_2 = x_3 = 0$, and $x_1 = x_4 = 1/(a_1 - a_4)$.

**Problem 6**
**Solution** If not, then considering ALM we have 4.AL.AM.sin A > AB.AC.sin A, so 4.AL.AM > AB.AC = (AM + BM)(AL + CL), so 3.AL.AM > AM.CL + BM.AL + BM.CL. Set *k* = BK/CK, *l* = CL/AL, *m* = AM/BM, and this inequality becomes:
3 > *l* + 1/*m* + *l*/*m*.
Similarly, considering the other two triangles we get: 3 > *k* + 1/*l* + *k*/*l*, and 3 > *m* + 1/*k* + *m*/*k*.
Adding gives: 9 > *k* + *l* + *m* + 1/*k* + 1/*l* + 1/*m* + *k*/*l* + *l*/*m* + *m*/*k*, which is false by the arithmetic/geometric mean inequality.

## 1967

**Problem 1**
**Solution** Evidently the parallelogram is a red herring, since the circles cover it iff and only if the three circles center A, B, D cover the triangle ABD.
The three circles radius x and centers the three vertices cover an acute-angled triangle ABC iff x is at least R, the circumradius. The circumcenter O is a distance R from each vertex, so the condition is clearly necessary. If the perpendiculars from O to AB, BC, CA are OP, OQ, OR, then the circle center A, radius R covers the quadrilateral APOR, the circle center B, radius R covers the quadrilateral BPOQ, and the circle center C radius R covers the quadrilateral CQOR, so the condition is also sufficient.
We need an expression for R in terms of a and A. We can express BD two ways: 2R sin A, and $\sqrt{(a^2 + 1 - 2a \cos A)}$. So a necessary and sufficient condition for the covering is 4 $\sin^2$A >= ($a^2$ + 1 - 2a cos A), which reduces to a <= cos A + √3 sin A, since cos A <= a (the foot of the perpendicular from D onto AB must lie between A and B).

**Problem 2**
**Solution** Let the tetrahedron be ABCD and assume that all edges except AB have length at most 1. The volume is the 1/3 x area BCD x height of A above BCD. The height is at most the height of A above CD, so we maximise the volume by taking the triangles ACD and BCD to be perpendicular. Then the volume is 1/6 x CD x altitude from A to CD x altitude from B to CD. If AC or AD is less than 1, then we can increase the altitude from A to CD whilst keeping CD fixed by taking AC = AD = 1. Similarly for the altitude from B to CD. So if CD = z, the volume is z(1 - $z^2$/4)/6. This is slightly awkward to maximise without using calculus. Factorise as: z(1 - z/2)(1 + z/2)/6 <= z(1 - z/2)(1 + 1/2)/6 = z/4 - $z^2$/8 = 1/8 - $(z - 1)^2$/8 <= 1/8. Checking back, we find that z = 1 gives z(1 - $z^2$/4)/6 = 1/8, so that is indeed the maximum value.

**Problem 3**
**Solution** The key is that $c_a - c_b$ = (a - b)(a + b + 1). Hence the product ($c_{m+1} - c_k$)($c_{m+2}$ -

$c_k$) ... ($c_{m+n}$ - $c_k$) is the product of the n consecutive numbers (m - k + 1), ... , (m - k + n), times the product of the n consecutive numbers (m + k + 2), ... , (m + k + n + 1). The first product is just the binomial coefficient (m-k+n)Cn times n!, so it is divisible by n!. The second product is 1/(m + k + 1) x (m + k + 1)(m + k + 2) ... (m + k + n + 1) = 1/(m + k + 1) x (m+k+n+1)C(n+1) x (n+1)!. But m + k + 1 is a prime greater than n + 1, so it has no factors in common with (n+1)!, hence the second product is divisible by (n+1)!. Finally note that $c_1c_2$ ... $c_n$= n! (n+1)!.

**Problem 4**
**Solution** Take any triangle similar to $A_1B_1C_1$ and circumscribing $A_0B_0C_0$. For example, take an arbitary line through $A_0$ and then lines through $B_0$ and $C_0$ at the appropriate angles to the first line. Label the triangle's vertices X, Y, Z so that $A_0$ lies on YZ, $B_0$ on ZX, and $C_0$ on XY. Now any circumscribed ABC (labeled with the same convention) must have C on the circle through $A_0$, $B_0$ and Z, because it has angle C = angle Z = angle $C_1$. Similarly it must have B on the circle through $C_0$, $A_0$ and Y, and it must have A on the circle through $B_0$, $C_0$ and X.
Consider the side AB. It passes through $C_0$. Its length is twice the projection of the line joining the centers of the two circles onto AB (because each center projects onto the midpoint of the part of AB that is a chord of its circle). But this projection is maximum when it is parallel to the line joining the two centers. The area is maximised when AB is maximised (because all the triangles are similar), so we take AB parallel to the line joining the centers. [Note, in passing, that this proves that the other sides must also be parallel to the lines joining the respective centers and hence that the three centers form a triangle similar to $A_1B_1C_1$.]

**Problem 5**
**Solution** Take $|a_1|$ >= $|a_2|$ >= ... >= $|a_8|$. Suppose that $|a_1|$, ... , $|a_r|$ are all equal and greater than $|a_{r+1}|$. Then for sufficiently large n, we can ensure that $|a_s|^n$ < 1/8 $|a_1|^n$ for s > r, and hence the sum of $|a_s|^n$ for all s > r is less than $|a_1|^n$. Hence r must be even with half of $a_1$, ... , $a_r$ positive and half negative.
If that does not exhaust the $a_i$, then in a similar way there must be an even number of $a_i$ with the next largest value of $|a_i|$, with half positive and half negative, and so on. Thus we find that $c_n$ = 0 for all odd n.

**Problem 6**
**Solution** Let the number of medals remaining at the start of day r be $m_r$. Then $m_1$ = m, and 6($m_k$ - k)/7 = $m_{k+1}$ for k < n with $m_n$ = n.
After a little rearrangement, we find that m = 1 + 2(7/6) + 3$(7/6)^2$ + ... + n$(7/6)^{n-1}$.
Summing, we get m = 36(1 - (n + 1)$(7/6)^n$ + n $(7/6)^{n+1}$) = 36 + (n - 6)$7^n/6^{n-1}$. 6 and 7 are coprime, so $6^{n-1}$ must divide n - 6. But $6^{n-1}$ > n - 6, so n = 6 and m = 36.

## 1968

**Problem 1**
**Solution** Let the sides be a, a+1, a+2, the angle oppose a be A, the angle opposite a+1 be B, and the angle opposite a+2 be C.





Using the cosine rule, we find cos A = (a+5)/(2a+4), cos B = (a+1)/2a, cos C = (a-3)/2a.
Finally, using cos 2x = 2 $\cos^2$x - 1, we find solutions a = 4 for C = 2A, a = 1 for B = 2A, and no solutions for C = 2B.
a = 1 is a degenerate solution (the triangle has the three vertices collinear). The other solution is 4, 5, 6.

**Problem 2**
**Solution** Suppose n has m > 1 digits. Let the first digit be d. Then the product of the digits is at most d.$9^{m-1}$ < d.$10^{m-1}$ <= n. But ($n^2$ - 10n - 22) - n = n(n - 11) - 22 > 0 for n >= 13. So there are no solutions for n > = 13. But $n^2$ - 10n - 22 < 0 for n <= 11, so the only possible solution is n=12 and indeed that is a solution.

**Problem 3**
**Solution** Let f(x) = $ax^2$ + bx + c - x. Then f(x)/a = $(x + (b-1)/2a)^2$ + (4ac - $(b-1)^2$)/$4a^2$.
Hence if 4ac - $(b-1)^2$ > 0, then f(x) has the same sign for all x. But f(x) > 0 means $ax^2$ + bx + c > x, so if {$x_i$} is a solution, then either $x_1 < x_2 < ... < x_n < x_1$, or $x_1 > x_2 > ... > x_n > x_1$. Either way we have a contradiction. So if 4ac - $(b-1)^2$ > 0 there cannot be any solutions.
If 4ac - $(b-1)^2$ = 0, then we can argue in the same way that either $x_1 <= x_2 <= ... <= x_n <= x_1$, or $x_1 >= x_2 >= ... >= x_n >= x_1$. So we must have all $x_i$ = the single root of f(x) = 0 (which clearly is a solution).
If 4ac - $(b-1)^2$ < 0, then f(x) = 0 has two distinct real roots y and z and so we have at least two solutions to the equations: all $x_i$ =y, and all $x_i$ = z. We may, however, have additional solutions. For example, if a = 1, b = 0, c = -1 and n is even, then we have the additional solution $x_1 = x_3 = x_5 = ... = 0$, $x_2 = x_4 = ... = -1$.

**Problem 4**
**Solution** The trick is to consider the longest side. That avoids getting into lots of different possible cases for which edge is longer than the sum of the other two.
So assume the result is false and let AB be the longest side. Then we have AB > AC + AD and BA > BC + BD. So 2AB > AC + AD + BC + BC. But by the triangle inequality, AB < AC + CB, AB < AD + DB, so 2AB < AC + CB + AD + DB. Contradiction.

**Problem 5**
**Solution** Directly from the equality given: f(x+a) >= 1/2 for all x, and hence f(x) >= 1/2 for all x. So f(x+2a) = 1/2 + √(f(x+a) - $f(x+a)^2$) = 1/2 + √(f(x+a) (1 - f(x+a)) = 1/2 + √(1/4 - f(x) + $f(x)^2$) = 1/2 + (f(x) - 1/2) = f(x). So f is periodic with period 2a.
We may take f(x) to be arbitary in the interval [0,1). For example, let f(x) = 1 for 0 <= x < 1, f(x) = 1/2 for 1 <= x < 2. Then use f(x+2) = f(x) to define f(x) for all other values of x.

**Problem 6**
**Solution** For any real x we have [x] = [x/2] + [(x+1]/2]. For if x = 2n + 1 + k, where n is an integer and 0 <= k < 1, then lhs = 2n + 1, and rhs = n + n + 1. Similarly, if x = 2n + k.
Hence for any integer n, we have: [$n/2^k$] - [$n/2^{k+1}$] = [($n/2^k$ + 1)/2] = [(n + $2^k$)/$2^{k+1}$].
Hence summing over k, and using the fact that n < $2^k$ for sufficiently large k, so that [$n/2^k$ ] = 0, we have: n = [(n + 1)/2] + [(n + 2)/4] + [(n + 4)/8] + ... .

## 1969

**Problem 1**
**Solution** $n^4 + 4 r^4 = (n^2 + 2rn + 2r^2)(n^2 - 2rn + 2r^2)$. Clearly the first factor is greater than 1, the second factor is $(n - r)^2 + r^2$, which is also greater than 1 for r greater than 1. So we may take $m = 4 r^4$ for any r greater than 1.

**Problem 2**
**Solution** f is not identically zero, because $f(-a_1) = 1 + 1/2 \cos(a_2 - a_1) + ... > 1 - 1/2 - 1/4 - ... - 1/2^{n-1} > 0$.
Using the expression for $\cos(x + y)$ we obtain $f(x) = b \cos x + c \sin x$, where $b = \cos a_1 + 1/2 \cos a_2 + ... + 1/2^{n-1} \cos a_n$, and $c = - \sin a_1 - 1/2 \sin a_2 - ... - 1/2^{n-1} \sin a_n$. b and c are not both zero, since f is not identically zero, so $f(x) = \sqrt{(b^2 + c^2)} \cos(d + x)$, where $\cos d = b/\sqrt{(b^2 + c^2)}$, and $\sin d = c/\sqrt{(b^2 + c^2)}$. Hence the roots of $f(x) = 0$ are just m.pi + pi/2 - d.

**Problem 3**
**Solution** A plodding question. Take the tetrahedron to be ABCD.
Take k = 1 and AB to have length a, the other edges length 1. Then we can hinge triangles ACD and BCD about CD to vary AB. The extreme values evidently occur with A, B, C, D coplanar. The least value, 0, when A coincides with B, and the greatest value $\sqrt{3}$, when A and B are on opposite sides of CD. We rule out the extreme values on the grounds that the tetrahedron is degenerate, thus obtaining $0 < a < \sqrt{3}$.
For k = 5, the same argument shows that $0 < 1 < \sqrt{3}\, a$, and hence $a > 1/\sqrt{3}$.
For k = 2, there are two possible configurations: the sides length a adjacent, or not. Consider first the adjacent case. Take the sides length a to be AC and AD. As before, the two extreme cases gave A, B, C, D coplanar. If A and B are on opposite sides of CD, then $a = \sqrt{(2 - \sqrt{3})}$. If they are on the same side, then $a = \sqrt{(2 + \sqrt{3})}$. So this configuration allows any a satisfying $\sqrt{(2 - \sqrt{3})} < a < \sqrt{(2 + \sqrt{3})}$.
The other configuration has AB = CD = a. One extreme case has a = 0. We can increase a until we reach the other extreme case with ADBC a square side 1, giving $a = \sqrt{2}$. So this configuration allows any a satisfying $0 < a < \sqrt{2}$. Together, the two configurations allow any a satisfying: $0 < a < \sqrt{(2 + \sqrt{3})}$.
This also solves the case k = 4, and allows any a satisfying: $a > 1/\sqrt{(2 + \sqrt{3})} = \sqrt{(2 - \sqrt{3})}$.
For k = 3, any value of a > 0 is allowed. For a <= 1, we may take the edges length a to form a triangle. For a >= 1 we may take a triangle with unit edges and the edges joining the vertices to the fourth vertex to have length a.

**Problem 4**
**Solution** Let the three centers be $O_1$, $O_2$ and $O_3$. We show that $O_1$ is the midpoint of $O_2O_3$. In fact it is sufficient to show that $O_1$ lies on $O_2O_3$, because then we can reflect the known tangent AB in the line $O_2O_3$.
As usual, let AB = c, BC = a, CA = b. Let the in-circle touch AB at P, AC at Q and BC at R. Then since angle ACB = 90, $O_1QCR$ is a square. Also AQ = AP and BP = BR, so $r_1 = b - AP$, and $r_1 = a - BP = a - (c - AP)$. Adding: $r_1 = (a + b - c)/2$, and $AP = (b + c - a)/2$.
Let the circle center $O_2$ touch AB at X, and the circle center $O_3$ touch AB at Y. Let O be the midpoint of AB. Now consider the right-angled triangle $OXO_2$. Since the circle





center $O_2$ touches the semicircle, $OO_2 = c/2 - r_2$. $OX = OD + DX = (c/2 - AD) + r_2$. Also, by similar triangles, $AD = b^2/c$. So, using Pythagoras: $(c/2 - r_2)^2 = r_2^2 + (c/2 - b^2/c + r_2)^2$. Multiplying out and rearranging: $r_2^2 - 2r_2(c - b^2/c) - (b^2 - b^4/c^2)$. But ABC is right-angled, so $c^2 = a^2 + b^2$, and hence $c - b^2/c = a^2/c$ and $b^2 - b^4/c^2 = a^2b^2/c^2$. So $r_2^2 + 2r_2\, a^2/c - a^2b^2/c^2 = 0$, which has roots $r_2 = a - a^2/c$ (positive) and $- a + a^2/c$ (negative). So $r_2 = a - a^2/c$. Similarly, $r_3 = b - b^2/c$. So $O_2X + O_3Y = XY = r_2 + r_3 = a + b - c = 2 r_1$.
$XP = AP - AX = AP - (AD - DX) = (b + c - a)/2 - (b^2/c - r_2) = (b + c - a)/2 - (c - a) = (a + b - c)/2 = r_1$. We now have all we need: $XP = PY = PO_1$, and $XO_2 + YO_3 = 2 PO_1$.

**Problem 5**
**Solution** $(n-3)(n-4)/2$ is a poor lower bound.
Observe first that any 5 points include 4 forming a convex quadrilateral. For take the convex hull. If it consists of more than 3 points, we are done. If not, it must consist of 3 points, A, B and C, with the other 2 points, D and E, inside the triangle ABC. Two vertices of the triangle must lie on the same side of the line DE and they form convex quadrilateral with D and E.
Given n points, we can choose 5 in $n(n-1)(n-2)(n-3)(n-4)/120$ different ways. Each choice gives us a convex quadrilateral, but any given convex quadrilateral may arise from n-4 different sets of 5 points, so we have at least $n(n-1)(n-2)(n-3)/120$ different convex quadrilaterals. We now show that $n(n-1)(n-2)(n-3)/120 >= (n-3)(n-4)/2$ for all n >= 5.
We wish to prove that $n(n-1)(n-2) >= 60(n-4)$, or $n(n-1)(n-2) - 60(n-4) >= 0$. Trial shows equality for n = 5 and 6, so we can factorise and get $(n-5)(n-6)(n+8)$, which is clearly at least 0 for n at least 5.

**Problem 6**
**Solution** Let $a_1 = x_1y_1 - z_1^2$ and $a_2 = x_2y_2 - z_2^2$. We apply the arithmetic/geometric mean result 3 times:
(1) to $a_1^2$, $a_2^2$, giving $2a_1a_2 <= a_1^2 + a_2^2$;
(2) to $a_1$, $a_2$, giving $\sqrt{(a_1a_2)} <= (a_1 + a_2)/2$;
(3) to $a_1y_2/y_1$, $a_2y_1/y_2$, giving $\sqrt{(a_1a_2)} <= (a_1y_2/y_1 + a_2y_1/y_2)/2$;
We also use $(z_1/y_1 - z_2/y_2)^2 >= 0$. Now $x_1y_1 > z_1^2 >= 0$, and $x_1 > 0$, so $y_1 > 0$. Similarly, $y_2 > 0$. So:
(4) $y_1y_2(z_1/y_1 - z_2/y_2)^2 >= 0$, and hence $z_1^2y_2/y_1 + z_2^2y_1/y_2 >= 2z_1z_2$.
Using (3) and (4) gives $2\sqrt{(a_1a_2)} <= (x_1y_2 + x_2y_1) - (z_1^2y_2/y_1 + z_2^2y_1/y_2) <= (x_1y_2 + x_2y_1 - 2z_1z_2)$.
Multiplying by (2) gives: $4a_1a_2 <= (a_1 + a_2)(x_1y_2 + x_2y_1 - 2z_1z_2)$.
Adding (1) and $2a_1a_2$ gives: $8a_1a_2 <= (a_1 + a_2)^2 + (a_1 + a_2)(x_1y_2 + x_2y_1 - 2z_1z_2) = a(a_1 + a_2)$, where $a = (x_1 + x_2)(y_1 + y_2) - (z_1 + z_2)^2$. Dividing by $a_1a_2a$ gives the required inequality.
Equality requires $a_1 = a_2$ from (1), $y_1 = y_2$ from (2), $z_1 = z_2$ from (3), and hence $x_1 = x_2$. Conversely, it is easy to see that these conditions are sufficient for equality.

## 1970

**Problem 1**
**Solution** We need an expression for r/q. There are two expressions, one in terms of angles and the other in terms of sides. The latter is a poor choice, because it is both harder to derive and less useful. So we derive the angle expression.

Let O be the center of the in-circle for ABC and X the center of the external circle for ABC. O is the intersection of the two angle bisectors from A and B, so c = r(cot A/2 + cot B/2). The X lies on the bisector of the external angle, so angle XAB is 90 - A/2. Similarly, angle XBA is 90 - B/2, so c = q(tan A/2 + tan B/2). Hence r/q = (tan A/2 + tan B/2)/(cot A/2 + cot B/2) = tan A/2 tan B/2.
Applying this to the other two triangles, we get $r_1/q_1$ = tan A/2 tan CMA/2, $r_2/q_2$ = tan B/2 tan CMB/2. But CMB/2 = 90 - CMA/2, so tan CMB/2 = 1/tan CMA/2. Hence result.

**Problem 2**
**Solution** We have $a^nb^m > b^na^m$ for n > m. Hence $a^nB' > b^nA'$. Adding $a^nb^n$ to both sides gives $a^nB > b^nA$. Hence $x_na^nB > x_nb^nA$. But $x_na^n$ = A - A' and $x_nb^n$ = B - B', so (A - A')B > (B - B')A. Hence result.
Note that the only purpose of requiring $x_{n-1} > 0$ is to prevent A' and B' being zero.

**Problem 3**
**Solution** (a) Each term of the sum is non-negative, so $b_n$ is non-negative. Let $c_k = \sqrt{a_k}$. Then the kth term = $(1 - a_{k-1}/a_k)/\sqrt{a_k} = c_{k-1}^2/c_k\ (1/a_{k-1} - 1/a_k) = c_{k-1}^2/c_k\ (1/c_{k-1} + 1/c_k)(1/c_{k-1} - 1/c_k)$. But $c_{k-1}^2/c_k\ (1/c_{k-1} + 1/c_k) <= 2$, so the kth term $<= 2(1/c_{k-1} - 1/c_k)$. Hence $b_n <= 2 - 2/c_n < 2$.
(b) Let $c_k = d^k$, where d is a constant > 1, which we will choose later. Then the kth term is $(1 - 1/d^2)1/d^k$, so $b_n = (1 - 1/d^2)(1 - 1/d^{n+1})/(1 - 1/d) = (1 + 1/d)(1 - 1/d^{n+1})$. Now take d sufficiently close to 1 that 1 + 1/d > c, and then take n sufficiently large so that $(1 + 1/d)(1 - 1/d^{n+1}) > c$.

**Problem 4**
**Solution** The only primes dividing numbers in the set can be 2, 3 or 5, because if any larger prime was a factor, then it would only divide one number in the set and hence only one product. Three of the numbers must be odd. At most one of the odd numbers can be a multiple of 3 and at most one can be a multiple of 5. The other odd number cannot have any prime factors. Contradiction. So there are no such sets.

**Problem 5**
**Solution** The first step is to show that angles ADB and ADC are also 90. Let H be the intersection of the altitudes of ABC and let CH meet AB at X. Planes CED and ABC are perpendicular and AB is perpendicular to the line of intersection CE. Hence AB is perpendicular to the plane CDE and hence to ED. So $BD^2 = DE^2 + BE^2$. Also $CB^2 = CE^2 + BE^2$. Subtracting: $CB^2 - BD^2 = CE^2 - DE^2$. But $CB^2 - BD^2 = CD^2$, so $CE^2 = CD^2 + DE^2$, so angle CDE = 90. But angle CDB = 90, so CD is perpendicular to the plane DAB, and hence angle CDA = 90. Similarly, angle ADB = 90.
Hence $AB^2 + BC^2 + CA^2 = 2(DA^2 + DB^2 + DC^2)$. But now we are done, because Cauchy's inequality gives $(AB + BC + CA)^2 <= 3(AB^2 + BC^2 + CA^2)$.
We have equality iff we have equality in Cauchy's inequality, which means AB = BC = CA.

**Problem 6**
**Solution**
At most 3 of the triangles formed by 4 points can be acute. It follows that at most 7 out of the 10 triangles formed by any 5 points can be acute. For given 10 points, the





maximum no. of acute triangles is: the no. of subsets of 4 points x 3/the no. of subsets of 4 points containing 3 given points. The total no. of triangles is the same expression with the first 3 replaced by 4. Hence at most 3/4 of the 10, or 7.5, can be acute, and hence at most 7 can be acute.
The same argument now extends the result to 100 points. The maximum number of acute triangles formed by 100 points is: the no. of subsets of 5 points x 7/the no. of subsets of 5 points containing 3 given points. The total no. of triangles is the same expression with 7 replaced by 10. Hence at most 7/10 of the triangles are acute.

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## 1971

**Problem 1**
**Solution** Take $a_1 < 0$, and the remaining $a_i = 0$. Then $E_n = a_1^{n-1} < 0$ for n even, so the proposition is false for even n.
Suppose n >= 7 and odd. Take any c > a > b, and let $a_1 = a$, $a_2 = a_3 = a_4 = b$, and $a_5 = a_6 = \dots = a_n = c$. Then $E_n = (a - b)^3(a - c)^{n-4} < 0$. So the proposition is false for odd n >= 7.
Assume $a_1 >= a_2 >= a_3$. Then in $E_3$ the sum of the first two terms is non-negative, because $(a_1 - a_3) >= (a_2 - a_3)$. The last term is also non-negative. Hence $E_3 >= 0$, and the proposition is true for n = 3.
It remains to prove $S_5$. Suppose $a_1 >= a_2 >= a_3 >= a_4 >= a_5$. Then the sum of the first two terms in $E_5$ is $(a_1 - a_2)\{(a_1 - a_3)(a_1 - a_4)(a_1 - a_5) - (a_2 - a_3)(a_2 - a_4)(a_2 - a_5)\} >= 0$. The third term is non-negative (the first two factors are non-positive and the last two non-negative). The sum of the last two terms is: $(a_4 - a_5)\{(a_1 - a_5)(a_2 - a_5)(a_3 - a_5) - (a_1 - a_4)(a_2 - a_4)(a_3 - a_4)\} >= 0$. Hence $E_5 >= 0$.

**Problem 2**
**Solution** The result is false for 8 vertices - for example, the cube. We get 8 cubes, with only faces in common, forming a cube 8 times as large.
This suggests a trick. Each $P_i$ is contained in D, the polyhedron formed from $P_1$ by doubling the scale. Take $A_1$ as the origin and take the vertex $B_i$ to have twice the coordinates of $A_i$. Given a point X inside $P_1$, the midpoint of $P_iX$ must lie in $P_1$ by convexity. Hence the point with doubled coordinates, which is obtained by adding the coordinates of $A_i$ to the coordinates of X, lies in D. In other words every point of $P_i$ lies in D. But the volume of D is 8 times the volume of $P_1$, which is less than the sum of the volumes of $P_1, \dots , P_9$.

**Problem 3**
**Solution** We show how to enlarge a set of r such integers to a set of r+1. So suppose $2^{n_1} - 3, \dots , 2^{n_r} - 3$ are all relatively prime. The idea is to find $2^n - 1$ divisible by $m = (2^{n_1} - 3) \dots (2^{n_r} - 3)$, because then $2^n - 3$ must be relatively prime to all of the factors of m. At least two of $2^0, 2^1, \dots , 2^m$ must be congruent mod m. So suppose $m_1 > m_2$ and $2^{m_1} \equiv 2^{m_2}$ (mod m), then we must have $2^{m_1 - m_2} - 1 \equiv 0$ (mod m), since m is odd. So we may take $n_{r+1}$ to be $m_1 - m_2$.

**Problem 4**
**Solution** The key is to pretend the tetrahedron is made of cardboard, cut it along three edges and unfold it. Suppose we do this to get the hexagon CAC'BDB'. Now the path is a line joining Y on B'C to Y' on the opposite side BC' of the hexagon. Clearly this line

must be straight for a minimal path. If B'C and BC' are parallel, then we can take Y anywhere on the side and the minimal path length is the expression given.
But if they are not parallel, then the minimal path will come from an extreme position. Suppose CC' < BB'. If the interior angle CAC' is less than 180, then the minimal path is obtained by taking Y at C. But this does not meet the requirement that Y be an interior point of the edge, so there is no minimal path in the permitted set. If the interior angle CAC' is greater than 180, then the minimal path is obtained by taking X and T at A. Again this is not permitted.
The problem therefore reduces to finding the condition for B'C and BC' to be parallel. This is evidently angles BCD + DCA + CAD + BAD + BAC + ACB = 360. But DCA + CAD = 180 - ADC, and BAC + ACB = 180 - ABC, so we obtain the condition given.

**Problem 5**
**Solution** Take $a_1, a_2, \ldots, a_m$ to be points a distance 1/2 from the origin O. Form the set of $2^m$ points $\pm a_1 \pm a_2 \pm \ldots \pm a_m$. Given such a point, it is at unit distance from the m points with just one coefficient different. So we are home, provided that we can choose the $a_i$ to avoid any other pairs of points being at unit distance, and to avoid any degeneracy (where some of the $2^m$ points coincide).
The distance between two points in the set is $|c_1a_1 + c_2a_2 + \ldots + c_ma_m|$, where $c_i = 0$, 2 or -2. So let us choose the $a_i$ inductively. Suppose we have already chosen up to m. The constraints on $a_{m+1}$ are that we do not have $|c_1a_1 + c_2a_2 + \ldots + c_ma_m + 2a_{m+1}|$ equal to 0 or 1 for any $c_i = 0$, 2 or -2, apart from the trivial cases of all $c_i = 0$. Each | | = 0 rules out a single point and each | | = 1 rules out a circle which intersects the circle radius 1/2 about the origin at 2 points and hence rules out two points. So the effect of the constraints is to rule out a finite number of points, whereas we have uncountably many to choose from.

**Problem 6**
**Solution** Let x be the smallest row or column sum. If x >= n/2, then we are done, so assume x < n/2. Suppose it is a row. (If not, interchange rows and columns.) The number of non-zero elements in the row, y, must also satisfy y < n/2, since each non-zero element is at least 1. Now move across this row summing the columns. The y columns with a non-zero element have sum at least x (by the definition of x). The n - y columns with a zero have sum at least n - x. Hence the total sum is at least $xy + (n - x)(n - y) = n^2/2 + (n - 2x)(n - 2y)/2 > n^2/2$.
The result is evidently best possible, because we can fill the matrix alternately with zeros and ones (so that $a_{ij} = 1$ if i and j are both odd or both even, 0 otherwise). For n even, every row and column has n/2 1s, so the condition is certainly satisfied and the total sum is $n^2/2$. For n odd, odd numbered rows have (n+1)/2 1s and even numbered one less. But the only zeros are in positions which have either the row or the column odd-numbered, so the sum in such cases is n as required. The total sum is $n^2/2 + 1/2$. Alternatively, for n even, we could place n/2 down the main diagonal.

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## 1972

**Problem 1**
**Solution** The number of non-empty subsets is $2^{10} - 1 = 1023$. The sum of each subset is at most 90 + ... + 99 = 945, so there must be two distinct subsets A and B with the same sum. A \ B and B \ A are disjoint subsets, also with the same sum.





**Problem 2**
**Solution** A little tinkering soon shows that it is easy to divide a cyclic quadrilateral ABCD into 4 cyclic quadrilaterals. Take a point P inside the quadrilateral and take an arbitary line PK joining it to AB. Now take L on BC so that angle KPL is 180 - angle B (thus ensuring that KPLB is cyclic), then M on CD so that angle LPM = 180 - angle C, then N on AD so that angle MPN is 180 - angle D. Then angle NPK is 180 - angle A. We may need to impose some restrictions on P and K to ensure that we can obtain the necessary angles. It is not clear, however, what to do next.
The trick is to notice that the problem is easy if two sides are parallel. For then we may take arbitarily many lines parallel to the parallel sides and divide the original quadrilateral into any number of parts.
So we need to arrange our choice of P and K so that one of the new quadrilaterals has parallel sides. But that is easy, since K is arbitary. So take PK parallel to AD, then we must also have PL parallel to CD.
It remains to consider how we ensure that the points lie on the correct sides. Consider first K and L. K cannot lie on AD since PK is parallel to AD, and we can avoid it lying on BC by taking P sufficiently close to D. Similarly, taking P sufficiently close to D ensures that L lies on BC. Now suppose that M and N are both on AD. Then if we keep K fixed and move P closer to CD, N will move on to CD, leaving M on AD.

**Problem 3**
**Solution** The trick is to find a recurrence relation for f(m,n) = (2m)!(2n)!/(m!n!(m+n)!). In fact, f(m,n) = 4 f(m,n-1) - f(m+1,n-1), which is sufficient to generate all the f(m,n), given that f(m,0) = (2m)!/(m!m!), which we know to be integeral.

**Problem 4**
**Solution** Answer: $x_1 = x_2 = x_3 = x_4 = x_5$.
The difficulty with this problem is that it has more information than we need. There is a neat solution in Greitzer which shows that all we need is the sum of the 5 inequalities, because one can rewrite that as $(x_1x_2 - x_1x_4)^2 + (x_2x_3 - x_2x_5)^2 + \ldots + (x_5x_1 - x_5x_3)^2 + (x_1x_3 - x_1x_5)^2 + \ldots + (x_5x_2 - x_5x_4)^2 <= 0$. The difficulty is how one ever dreams up such an idea!
The more plodding solution is to break the symmetry by taking $x_1$ as the largest. If the second largest is $x_2$, then the first inequality tells us that $x_1^2$ or $x_2^2 = x_3x_5$. But if $x_3$ and $x_5$ are unequal, then the larger would exceed $x_1$ or $x_2$. Contradiction. Hence $x_3 = x_5$ and also equals $x_2$ or $x_1$. If they equal $x_1$, then they would also equal $x_2$ (by definition of $x_2$), so in any case they must equal $x_2$. Now the second inequality gives $x_2 = x_1x_4$. So either all the numbers are equal, or $x_1 > x_2 = x_3 = x_5 > x_4$. But in the second case the last inequality is violated. So the only solution is all numbers equal.
If the second largest is $x_5$, then we can use the last inequality to deduce that $x_2 = x_4 = x_5$ and proceed as before.
If the second largest is $x_3$, then the fourth inequality gives that $x_1 = x_3 = x_5$ or $x_1 = x_3 = x_4$. In the first case, $x_5$ is the second largest and we are home already. In the second case, the third inequality gives $x_3^2 = x_2x_5$ and hence $x_3 = x_2 = x_5$ (or one of $x_2$, $x_5$ would be larger than the second largest). So $x_5$ is the second largest and we are home.
Finally, if the second largest is $x_4$, then the second inequality gives $x_1 = x_2 = x_4$ or $x_1 = x_3 = x_4$. Either way, we have a case already covered and so the numbers are all equal.

**Problem 5**
**Solution** Let k be the least upper bound for |f(x)|. Suppose |g(y)| > 1. Take any x with

$|f(x)| > 0$, then $2k >= |f(x+y)| + |f(x-y)| >= |f(x+y) + f(x-y)| = 2|g(y)||f(x)|$, so $|f(x)| < k/|g(y)|$. In other words, $k/|g(y)|$ is an upper bound for $|f(x)|$ which is less than k. Contradiction.

**Problem 6**
**Solution** Intuitively, we can place A and B on the two outer planes with AB perpendicular to the planes. Then tilt AB in one direction until we bring C onto one of the middle planes (keeping A and B on the outer planes), then tilt AB the other way (keeping A, B, C on their respective planes) until D gets onto the last plane.
Take A as the origin. Let the vectors AB, AC, AD be $\underline{b}$, $\underline{c}$, $\underline{d}$. Take p as one of the outer planes. Let the distances to the other planes be e, f, g. Now we find a vector $\underline{n}$ satisfying: $\underline{n}.\underline{b} = e$, $\underline{n}.\underline{c} = f$, $\underline{n}.\underline{d} = g$. This is a system of three equations in three unknowns with non-zero determinant (because $\underline{b}.\underline{c} \times \underline{d}$ is non-zero), so it has a solution $\underline{n}$. Scale the tetrahedron by $|\underline{n}|$, orient p perpendicular to $\underline{n}/|\underline{n}|$, then B, C, D will be on the other planes as required.

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## 1973

**Problem 1**
**Solution** We proceed by induction on n. It is clearly true for n = 1. Assume it is true for 2n-1. Given $OP_i$ for 2n+1, reorder them so that all $OP_i$ lie between $OP_{2n}$ and $OP_{2n+1}$. Then $u = OP_{2n} + OP_{2n+1}$ lies along the angle bisector of angle $P_{2n}OP_{2n+1}$ and hence makes an angle less than 90 with $v = OP_1 + OP_2 + ... + OP_{2n-1}$ (which must lie between $OP_1$ and $OP_{2n-1}$ and hence between $OP_{2n}$ and $OP_{2n+1}$. By induction $|v| >= 1$. But $|u + v| >= |v|$ (use the cosine formula). Hence the result is true for 2n+1.
It is clearly best possible: take $OP_1 = ... = OP_n = -OP_{n+1} = ... = -OP_{2n}$, and $OP_{2n+1}$ in an arbitary direction.

**Problem 2**
**Solution** To warm up, we may notice that a regular hexagon is a planar set satisfying the condition.
Take two regular hexagons with a common long diagonal and their planes perpendicular. Now if we take A, B in the same hexagon, then we can find C, D in the same hexagon. If we take A in one and B in the other, then we may take C at the opposite end of a long diagonal from A, and D at the opposite end of a long diagonal from B.

**Problem 3**
**Solution** Put $y = x + 1/x$ and the equation becomes $y^2 + ay + b - 2 = 0$, which has solutions $y = -a/2 \pm\sqrt{(a^2 + 8 - 2b)}/2$. We require $|y| >= 2$ for the original equation to have a real root and hence we need $|a| + \sqrt{(a^2 + 8 - 4b)} >= 4$. Squaring gives $2|a| - b >= 2$. Hence $a^2 + b^2 >= a^2 + (2 - 2|a|)^2 = 5a^2 - 8|a| + 4 = 5(|a| - 4/5)^2 + 4/5$. So the least possible value of $a^2 + b^2$ is 4/5, achieved when a = 4/5, b = -2/5. In this case, the original equation is $x^4 + 4/5\ x^3 - 2/5\ x^2 + 4/5\ x + 1 = (x + 1)^2(x^2 - 6/5\ x + 1)$.

**Problem 4**
**Solution** In particular he must sweep the other two vertices. Let us take the triangle to be ABC, with side 1 and assume the soldier starts at A. So the path must intersect the circles radius $\sqrt{3}/4$ centered on the other two vertices. Let us look for the shortest path of this type. Suppose it intersects the circle center B at X and the circle center C at Y, and goes first to X and then to Y. Clearly the path from A to X must be a straight line and the path from X to Y must be a straight line. Moreover the shortest path from X to the circle center C follows the line XC and has length $AX + XC - \sqrt{3}/4$. So we are looking for the point X which minimises AX + XC.
Consider the point P where the altitude intersects the circle. By the usual reflection argument the distance AP + PC is shorter than the distance AP' + P'C for any other point P' on the line perpendicular to the altitude through P. Moreover for any point X on the circle, take AX to cut the line at P'. Then AX + XC > AP' + P'C > AP + PC.
It remains to check that the three circles center A, X, Y cover the triangle. In fact the circle center X covers the whole triangle except for a small portion near A and a small portion near C, which are covered by the triangles center A and Y.

**Problem 5**
**Solution** $f(x) = ax + b$ has fixed point $b/(1-a)$. If a = 1, then b must be 0, and any point is a fixed point. So suppose $f(x) = ax + b$ and $g(x) = ax + b'$ are in G. Then h the inverse of f is given by $h(x) = x/a - b/a$, and $hg(x) = x + b'/a - b/a$. This is in G, so we must have b' = b.
Suppose $f(x) = ax + b$, and $g(x) = cx + d$ are in G. Then $fg(x) = acx + (ad + b)$, and $gf(x) = acx + (bc + d)$. We must have $ad + b = bc + d$ and hence $b/(1-a) = c/(1-d)$, in other words f and g have the same fixed point.

**Problem 6**
**Solution** We notice that the constraints are linear, in the sense that if $b_i$ is a solution for $a_i$, q, and $b_i'$ is a solution for $a_i'$, q, then for any k, k' > 0 a solution for $ka_i + k'a_i'$, q is $kb_i + k'b_i'$. Also a "near" solution for $a_h = 1$, other $a_i = 0$ is $b_1 = q^{h-1}$, $b_2 = q^{h-2}$, ... , $b_{h-1} = q$, $b_h = 1$, $b_{h+1} = q$, ... , $b_n = q^{n-h}$. "Near" because the inequalities in (a) and (b) are not strict. However, we might reasonably hope that the inequalities would become strict in the linear combination, and indeed that is true. Define $b_r = q^{r-1}a_1 + q^{r-2}a_2 + ... + qa_{r-1} + a_r + qa_{r+1} + ... + q^{n-r}a_n$. Then we may easily verify that (a) - (c) hold.

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## 1974

**Problem 1**
**Solution** The player with 9 counters.
The total of the scores, 39, must equal the number of rounds times the total of the cards. But 39 has no factors except 1, 3, 13 and 39, the total of the cards must be at least 1 + 2 + 3 = 6, and the number of rounds is at least 2. Hence there were 3 rounds and the cards total 13.
The highest score was 20, so the highest card is at least 7. The score of 10 included at least one highest card, so the highest card is at most 8. The lowest card is at most 2, because if it was higher then the highest card would be at most 13 - 3 - 4 = 6, whereas we know it is at least 7. Thus the possibilities for the cards are: 2, 3, 8; 2, 4, 7; 1, 4, 8; 1, 5, 7. But the only one of these that allows a score of 20 is 1, 4, 8. Thus the scores were made up: 8 + 8 + 4 = 20, 8 + 1 + 1 = 10, 4 + 4 + 1 = 9. The last round must have been 4 to the player with 20, 8 to the player with 10 and 1 to the player with 9. Hence on each of the other two rounds the cards must have been 8 to the player with 20, 1 to the player with 10 and 4 to the player with 9.

**Problem 2**
**Solution** Extend CD to meet the circumcircle of ABC at E. Then CD.DE = AD.DB, so CD is the geometric mean of AD and DB iff CD = DE. So we can find such a point iff the distance of C from AB is less than the distance of AB from the furthest point of the arc AB on the opposite side of AB to C. The furthest point F is evidently the midpoint of the arc AB. F lies on the angle bisector of C. So angle FAB = angle FAC = C/2. Hence distance of F from AB is c/2 tan C/2 (as usual we set c = AB, b = CA, a = BC). The distance of C from AB is a sin B. So a necessary and sufficient condition is c/2 tan C/2 >= a sin B. But by the sine rule, a = c sin A/sin C, so the condition becomes (sin C/2 sin C)/(2 cos C/2) >= sin A sin B. But sin C = 2 sin C/2 cos C/2, so we obtain the condition quoted in the question.

**Problem 3**
**Solution** Let $k = \sqrt{8}$. Then $(1 + k)^{2n+1} = a + bk$, where b is the sum given in the question. Similarly, $(1 - k)^{2n+1} = a - bk$. This looks like a dead end, because eliminating a gives an unhelpful expression for b. The trick is to multiply the two expressions to get $7^{2n+1} = 8b^2 - a^2$. This still looks unhelpful, but happens to work, because we soon find that $7^{2n+1} \equiv \pm 2 \pmod 5$. So if b was a multiple of 5 then we would have a square congruent to $\pm 2 \pmod 5$ which is impossible.

**Problem 4**
**Solution** The requirement that the number of black and white squares be equal is equivalent to requiring that the each rectangle has an even number of squares. 2 + 4 + 6 + 8 + 10 + 12 + 14 + 16 = 72 > 64, so p < 8. There are 5 possible divisions of 64 into 7 unequal even numbers: 2 + 4 + 6 + 8 + 10 + 12 + 22; 2 + 4 + 6 + 8 + 10 + 16 + 18; 2 + 4 + 6 + 8 + 12 + 14 + 18; 2 + 4 + 6 + 10 + 12 + 14 + 16. The first is ruled out because a rectangle with 22 squares would have more than 8 squares on its longest side. The others are all possible.

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2 2 2 2 2 2 2 4   2 2 2 2 2 2 2 2
2 2 2 2 2 2 2 4   2 2 2 2 2 2 2 2
1 1 1 1 1 5 5 4   1 1 1 1 1 1 5 5
1 1 1 1 1 5 5 4   1 1 1 1 1 1 5 5
1 1 1 1 1 5 5 4   1 1 1 1 1 1 5 5
1 1 1 1 1 6 6 4   3 3 3 3 3 7 6 6
3 3 3 3 3 6 6 4   3 3 3 3 3 7 6 6
3 3 3 3 3 7 7 4   4 4 4 4 4 4 4 4

2 2 2 2 2 2 2 7   1 1 1 1 1 1 1 1
2 2 2 2 2 2 2 7   1 1 1 1 1 1 1 1
1 1 1 1 1 1 4 4   2 2 2 2 2 2 2 7
1 1 1 1 1 1 4 4   2 2 2 2 2 2 2 7
1 1 1 1 1 1 4 4   3 3 3 3 3 3 6 6
3 3 3 3 3 3 4 4   3 3 3 3 3 3 6 6
3 3 3 3 3 3 6 6   4 4 4 4 4 5 5 5
5 5 5 5 5 5 6 6   4 4 4 4 4 5 5 5
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**Problem 5**
**Solution** We show first that the sum must lie between 1 and 2. If we replace each denominator by a+b+c+d then we reduce each term and get 1. Hence the sum is more than 1. Suppose a is the largest of the four reals. Then the first term is less than 1. The second and fourth terms have denominators greater than b+c+d, so the terms are





increased if we replace the denominators by b+c+d. But then the last three terms sum to 1. Thus the sum of the last three terms is less than 1. Hence the sum is less than 2.
If we set a = c = 1 and make b and d small, then the first and third terms can be made arbitarily close to 1 and the other two terms arbitarily close to 0, so we can make the sum arbitarily close to 2. If we set a = 1, c = d and make b and c/b arbitarily small, then the first term is arbitarily close to 1 and the last three terms are all arbitarily small, so we can make the sum arbitarily close to 1. Hence, by continuity, we can achieve any value in the open interval (1,2).

**Problem 6**
**Solution** Suppose that A(x) and B(x) are two polynomials with integer coefficients which are identical except for their constant terms, which differ by 2. Suppose A(r) = 0, and B(s) =0 with r and s integers. Then subtracting we get 2 plus a sum of terms $a(r^i - s^i)$. Each of these terms is divisible by (r - s), so 2 must be divisible by (r - s). Hence r and s differ by 0, 1 or 2.
Now let r be the smallest root of P(x) = 1 and P(x) = -1. The polynomial with r as a root can have at most d distinct roots and hence at most d distinct integer roots. If s is a root of the other equation then s must differ from r by 0, 1, or 2. But s >= r, so s = r, r+1 or r+2. Hence the other equation adds at most 2 distinct integer roots.

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## 1975

**Problem 1**
**Solution** If x >= x' and y >= y', then $(x - y)^2 + (x' - y')^2 \le (x - y')^2 + (x' - y)^2$. Hence if i < j, but $z_i \le z_j$, then swapping $z_i$ and $z_j$ reduces the sum of the squares. But we can return the order of the $z_i$ to $y_i$ by a sequence of swaps of this type: first swap 1 to the 1st place, then 2 to the 2nd place and so on.

**Problem 2**
**Solution** We must be able to find a set S of infinitely many $a_n$ in some residue class mod $a_i$. Take $a_j$ to be a member of S. Then for any $a_n$ in S satisfying $a_n > a_j$, we have $a_n = a_j$ + a multiple of $a_i$.

**Problem 3**
**Solution** Trigonometry provides a routine solution. Let BC = a, CA = b, AB = c. Then, by the sine rule applied to AQC, AQ = b/(2 sin 105) = b/(2 cos 15). Similarly, PB = a/(2 cos 15). Also AR = RB = c/(2 cos 15). So by the cosine rule $RP^2 = (a^2 + c^2 - 2ac \cos(B+60))/(4 \cos^2 15)$, and $RQ^2 = (b^2 + c^2 - 2bc \cos(A+60))/(4 \cos^2 15)$. So RP = RQ is equivalent to: $a^2 - 2ac \cos(60+B) = b^2 - 2bc \cos(60+A)$ and hence to $a^2 - ac \cos B + \sqrt{3}\, ac \sin B = b^2 - bc \cos A + \sqrt{3}\, bc \sin A$. By the sine rule, the sine terms cancel. Also b - b cos A = a cos C, and a - c cos B = b cos C, so the last equality is true and hence RP = RQ. We get an exactly similar expression for $PQ^2$ and show that it equals $2\ RP^2$ in the same way.
A more elegant solution is to construct S on the outside of AB so that ABS is equilateral. Then we find that CAS and QAR are similar and that CBS and PBR are similar. So QR/CS = PR/CS. The ratio of the sides is the same in each case (CA/QA = CB/PB since CQA and CPB are similar), so QR = PR. Also there is a 45 degree rotation between QAR and CAS and another 45 degree rotation between CBS and PBR, hence QR and PR are at 90 degrees.

**Problem 4**
**Solution** Let $X = 4444^{4444}$.Then X has less than 4.4444 = 17776 digits, so A is at most 9.17776 = 159984. Hence B is at most 6.9 = 54. But all these numbers are congruent mod 9. $4444 \equiv -2 \pmod 9$, so $X \equiv (-2)^{4444} \pmod 9$. But $(-2)^3 \equiv 1 \pmod 9$, and $4444 \equiv 1 \pmod 3$, so $X \equiv -2 \equiv 7 \pmod 9$. But any number less than 55 and congruent to 7 has digit sum 7 (possibilities are 7, 16, 25, 34, 43, 52). Hence the answer is 7.

**Problem 5**
**Solution** Let x be the angle $\cos^{-1}4/5$, so that cos x = 4/5, sin x = 3/5. Take points on the unit circle at angles 2nx for n integral. Then the distance between the points at angles 2nx and 2mx is 2 sin(n - m)x. The usual formula, giving sin(n - m)x in terms of sin x and cos x, shows that sin(n - m)x is rational. So it only remains to show that this process generates arbitarily many distinct points, in other words that x is not a rational multiple of pi.
This is quite hard. There is an elegant argument in sections 5 and 8 of Hadwiger et al, Combinatorial geometry in the Plane. But we can avoid it by observing that there are only finitely many numbers with are nth roots of unity for n <= 2 x 1975, whereas there are infinitely many Pythagorean triples, so we simply pick a triple which is not such a root of unity.

**Problem 6**
**Solution** (1) means that P is homogeneous of degree n for some n. Experimenting with low n, shows that the only solutions for n = 1, 2, 3 are (x - 2y), (x + y)(x - 2y), $(x + y)^2(x - 2y)$. It then obvious by inspection that $(x + y)^n(x - 2y)$ is a solution for any n.
Taking x = y = z in (2) shows that P(2x,x) = 0, so (x - 2y) is always a factor. Taking x = y = 1, z = -2 gives P(1,-1) $(2^n - 2) = 0$, so (x + y) is a factor for n > 1. All this suggests (but does not prove) that the general solution is $(x + y)^n(x - 2y)$.
Take y = 1 - x, z = 0 in (2) and we get: P(x, 1-x) = -1 - P(1-x, x). In particular, P(0,1) = -2. Now take z = 1 - x - y and we get: P(1-x, x) + P(1-y, y) + P(x+y, 1-x-y) = 0 and hence f(x+y) = f(x) + f(y), where f(x) = P(1-x, x) - 1. By induction we conclude that, for any integer m and real x, f(mx) = mf(x). Hence f(1/s) = 1/s f(1) and f(r/s) = r/s f(1) for any integers r, s. But P(0,1) = -2, so f(1) = -3. So f(x) = -3x for all rational x. But f is continuous, so f(x) = -3x for all x. So set x = b/(a+b), where a and b are arbitary reals (with a+b non-zero). Then $P(a,b) = (a+b)^nP(1-x, x) = (a+b)^n(-3b/(a+b) + 1) = (a+b)^{n-1}(a-2b)$, as claimed. [For a+b = 0, we appeal to continuity, or use the already derived fact that for n > 1, P(a,b) = 0.]

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## 1976

**Problem 1**
**Solution** At first sight, the length of the other diagonal appears unlikely to be significantly constrained. However, a little experimentation shows that it is hard to get such a low value as 16. This suggests that 16 may be the smallest possible value.
If the diagonal which is part of the 16 has length x, then the area is the sum of the areas of two triangles base x, which is xy/2, where y is the sum of the altitudes of the two triangles. y must be at most (16 - x), with equality only if the two triangles are right-angled. But $x(16 - x)/2 = (64 - (x - 8)^2)/2 <= 32$ with equality only iff x = 8. Thus the only way we can achieve the values given is with one diagonal length 8 and two sides





perpendicular to this diagonal with lengths totalling 8. But in this case the other diagonal has length $8\sqrt{2}$.

**Problem 2**
**Solution** We show that the graph of $P_n$ can be divided into $2^n$ lines each joining the top and bottom edges of the square side 4 centered on the origin (vertices (2,2), (-2,2), (-2,-2), (-2,2) ). We are then home because the upward sloping diagonal of the square, which represents the graph of y = x, must cut each of these lines and hence give $2^n$ distinct real roots of $P_n(x) = x$ in the range [-2,2]. But $P_n$ is a polynomial of degree $2^n$, so it has exactly $2^n$ roots. Hence all its roots are real and distinct.
We prove the result about the graph by induction. It is true for n = 1: the first line is the graph from x = -2 to 0, and the second line is the graph from 0 to 2. So suppose it is true for n. Then $P_1$ turns each of the $2^n$ lines for $P_n$ into two lines for $P_{n+1}$, so the result is true for n+1.
*Alternative solution:*
Put x = 2 cos t (so we are restricting attention to -2 <= x <= 2). Then we find $P_n(x) = 2 \cos 2^n t$, so the equation $P_n(x) = x$ becomes $\cos 2^n t = \cos t$. By inspection, has the $2^n$ solutions $t = 2k\pi/(2^n - 1)$ and $t = 2k\pi/(2^n + 1)$, giving $2^n$ distinct solutions in x.

**Problem 3**
**Solution** Answer: 2 x 3 x 5 or 2 x 5 x 6.
This is somewhat messy. The basic idea is that the sides cannot be too long, because then the ratio becomes too big. Let k denote the (real) cube root of 2. Given any integer n, let n' denote the least integer such that n'k <= n. Let the sides of the box be a <= b <= c. So we require 5a'b'c' = abc (*).
It is useful to derive n' for small n: 1' = 0, 2' = 1, 3' = 2, 4' = 3, 5' = 3, 6' = 4, 7' = 5, 8' = 6, 9' = 7, 10' = 7.
Clearly n'k >= n-2. But $6^3 > 0.4\ 8^3$, and hence $(n'k)^3 >= (n - 2)^3 > 0.4\ n^3$ for all n >= 8. We can check directly that $(n'k)^3 > 0.4\ n^3$ for n = 3, 4, 5, 6, 7. So we must have a = 2 (we cannot have a = 1, because 1' = 0).
From (*) we require b or c to be divisible by 5. Suppose we take it to be 5. Then since 5' = 3, the third side n must satisfy: n' = 2/3 n. We can easily check that 2k/3 < 6/7 and hence (2/3 nk + 1 ) < n for n >= 7, so n' > 2/3 n for n >= 7. This just leaves the values n = 3 and n = 6 to check (since n' = 2/3 n is integral so n must be a multiple of 3). Referring to the values above, both these work. So this gives us two possible boxes: 2 x 3 x 5 and 2 x 5 x 6.
The only remaining possibility is that the multiple of 5 is at least 10. But then it is easy to check that if it is m then m'/m >= 7/10. It follows from (*) that the third side r must satisfy r'/r <= 4/7. But using the limit above and referring to the small values above, this implies that r must be 2. So a = b = 2. But now c must satisfy c' = 4/5 c. However, that is impossible because 4/5 k > 1.

**Problem 4**
**Solution** Answer: $2 \times 3^{658}$.
There cannot be any integers larger than 4 in the maximal product, because for n > 4, we can replace n by 3 and n - 3 to get a larger product. There cannot be any 1s, because there must be an integer r > 1 (otherwise the product would be 1) and r + 1 > 1.r. We can also replace any 4s by two 2s leaving the product unchanged. Finally, there cannot be more than two 2s, because we can replace three 2s by two 3s to get a larger product.

Thus the product must consist of 3s, and either zero, one or two 2s. The number of 2s is determined by the remainder on dividing the number 1976 by 3.
1976 = 3.658 + 2, so there must be just one 2, giving the product 2 x $3^{658}$.

**Problem 5**
**Solution** We use a counting argument. If the modulus of each $x_i$ is at most n, then each of the linear combinations has a value between $-2n^2$ and $2n^2$, so there are at most $(4n^2 + 1)$ possible values for each linear combination and at most $(2n^2 + 1)^n$ possible sets of values. But there are 2n+1 values for each $x_i$ with modulus at most n, and hence $(2n+1)^{2n} = (4n^2+4n+1)^n$ sets of values. So two distinct sets must give the same set of values for the linear combinations. But now if these sets are $x_i$ and $x_i'$, then the values $x_i$-$x_i'$ give zero for each linear combination, and have modulus at most 2n. Moreover they are not all zero, since the two sets of values were distinct.

**Problem 6**
**Solution** Experience with recurrence relations suggests that the solution is probably the value given for $[u_n]$ plus its inverse. It is straightforward to verify this guess by induction.
Squaring $u_{n-1}$ gives the sum of positive power of 2, its inverse and 2. So $u_{n-1}$ - 2 = the sum of a positive power of 2 and its inverse. Multiplying this by $u_n$ gives a positive power of 2 + its inverse + 2 + 1/2, and we can check that the power of 2 is correct for $u_{n+1}$.

## 1977

**Problem 1**
**Solution** The most straightforward approach is to use coordinates. Take A, B, C, D to be (1,1), (-1,1), (-1,-1), (1,-1). Then K, L, M, N are (0, -2k), (2k, 0), (0, 2k), (-2k, 0), where k = (√3 - 1)/2. The midpoints of KL, LM, MN, NK are (k, -k), (k, k), (-k, k), (-k, -k). These are all a distance k√2 from the origin, at angles 315, 45, 135, 225 respectively. The midpoints of AK, BK, BL, CL, CM, DM, DN, AN are (h, j), (-h, j), (-j, h), (-j, -h), (-h, -j), (h, -j), (j, -h), (j, h), where h = 1/2, j = (1 - 1/2 √3). These are also at a distance k√2 from the origin, at angles 15, 165, 105, 255, 195, 345, 285, 75 respectively. For this we need to consider the right-angled triangle sides k, h, j. The angle x between h and k has sin x = j/k and cos x = h/k. So sin 2x = 2 sin x cos x = $2hj/k^2$ = 1/2. Hence x = 15.
So the 12 points are all at the same distance from the origin and at angles 15 + 30n, for n = 0, 1, 2, ... , 11. Hence they form a regular dodecagon.

**Problem 2**
**Solution** Answer: 16.
$x_1 + ... + x_7 < 0$, $x_8 + ... + x_{14} < 0$, so $x_1 + ... + x_{14} < 0$. But $x_4 + ... + x_{14} > 0$, so $x_1 + x_2 + x_3 < 0$. Also $x_5 + ... + x_{11} < 0$ and $x_1 + ... + x_{11} > 0$, so $x_4 > 0$. If there are 17 or more elements then the same argument shows that $x_5, x_6, x_7 > 0$. But $x_1 + ... + x_7 < 0$, and $x_5 + ... + x_{11} < 0$, whereas $x_1 + ... + x_{11} > 0$, so $x_5 + x_6 + x_7 < 0$. Contradiction.
If we assume that there is a solution for n = 16 and that the sum of 7 consecutive terms is -1 and that the sum of 11 consecutive terms is 1, then we can easily solve the equations to get: 5, 5, -13, 5, 5, 5, -13, 5, 5, -13, 5, 5, 5, -13, 5, 5 and we can check that this works for 16.





**Problem 3**
**Solution** Take a, b, c, d ≡ -1 (mod n). The idea is to take abcd which factorizes as ab.cd or ac.bd. The hope is that ab, cd, ac, bd will not factorize in $V_n$. But a little care is needed, since this is not necessarily true.
Try taking a = b = n - 1, c = d = 2n -1. $a^2$ must be indecomposable because it is less than the square of the smallest element in $V_n$. If ac = $2n^2$ - 3n + 1 is decomposable, then we have kk'n + k + k' = 2n - 3 for some k, k' >= 1. But neither of k or k' can be 2 or more, because then the lhs is too big, and k = k' = 1 does not work unless n = 5. Similarly, if $c^2$ is decomposable, then we have kk'n + k + k' = 4n - 4. k = k' = 1 only works for n = 2, but we are told n > 2. k = 1, k' = 2 does not work (it would require n = 7/2). k = 1, k' = 3 only works for n = 8. Other possibilities make the lhs too big.
So if n is not 5 or 8, then we can take the number to be $(n - 1)^2(2n - 1)^2$, which factors as $(n - 1)^2$ x $(2n - 1)^2$ or as (n - 1)(2n - 1) x (n - 1)(2n - 1). This does not work for 5 or 8: 16.81 = 36.36, but 36 decomposes as 6 x 6; 49.225 = 105.105, but 225 decomposes as 9 x 25.
For n = 5, we can use 3136 = 16.196 = 56.56. For n = 8, we can use 25921 = 49.529 = 161.161.

**Problem 4**
**Solution** Take y so that cos y = $a/\sqrt{(a^2 + b^2)}$, sin y = $b/\sqrt{(a^2 + b^2)}$, and z so that cos 2z = $A/\sqrt{(A^2 + B^2)}$, sin 2z = $B/\sqrt{(A^2 + B^2)}$. Then f(x) = 1 - c cos(x - y) - C cos2(x - z), where c = $\sqrt{(a^2 + b^2)}$, C = $\sqrt{(A^2 + B^2)}$.
f(z) + f(pi + z) >= 0 gives C <= 1. f(y + pi/4) + f(y - pi/4) >= 0 gives c <= √2.

**Problem 5**
**Solution** $a^2 + b^2$ >= $(a + b)^2/2$, so q >= (a + b)/2. Hence r < 2q. The largest square less than 1977 is 1936 = $44^2$. 1977 = $44^2$ + 41. The next largest gives 1977 = $43^2$ + 128. But 128 > 2.43. So we must have q = 44, r = 41. Hence $a^2 + b^2$ = 44(a + b) + 41, so $(a - 22)^2 + (b - 22)^2$ = 1009. By trial, we find that the only squares with sum 1009 are $28^2$ and $15^2$. This gives two solutions 50, 37 or 50, 7.

**Problem 6**
**Solution** The first step is to show that f(1) < f(2) < f(3) < ... . We do this by induction on n. We take $S_n$ to be the statement that f(n) is the unique smallest element of { f(n), f(n+1), f(n+2), ... }.
For m > 1, f(m) > f(s) where s = f(m-1), so f(m) is not the smallest member of the set {f(1), f(2), f(3), ... }. But the set is bounded below by zero, so it must have a smallest member. Hence the unique smallest member is f(1). So $S_1$ is true.
Suppose $S_n$ is true. Take m > n+1. Then m-1 > n, so by $S_n$, f(m-1) > f(n). But $S_n$ also tells us that f(n) > f(n-1) > ... > f(1), so f(n) >= n - 1 + f(1) >= n. Hence f(m-1) >= n+1. So f(m-1) belongs to { n+1, n+2, n+3, .. }. But we are given that f(m) > f(f(m-1)), so f(m) is not the smallest element of { f(n+1), f(n+2), f(n+3), ... }. But there must be a smallest element, so f(n+1) must be the unique smallest member, which establishes $S_{n+1}$. So, $S_n$ is true for all n.
So n <= m implies f(n) <= f(m). Suppose for some m, f(m) >= m+1, then f(f(m)) >= f(m+1). Contradiction. Hence f(m) <= m for all m. But since f(1) >=1 and f(m) > f(m-1) > ... > f(1), we also have f(m) >= m. Hence f(m) = m for all m.

## 1978

**Problem 1**
**Solution** We require $1978^m(1978^{n-m} - 1)$ to be a multiple of 1000=8.125. So we must have 8 divides $1978^m$, and hence m >= 3, and 125 divides $1978^{n-m} - 1$.
By Euler's theorem, $1978^{\phi(125)} \equiv 1 \pmod{125}$. $\phi(125) = 125 - 25 = 100$, so $1978^{100} \equiv 1 \pmod{125}$. Hence the smallest r such that $1978^r \equiv 1 \pmod{125}$ must be a divisor of 100 (because if it was not, then the remainder on dividing it into 100 would give a smaller r). That leaves 9 possibilities to check: 1, 2, 4, 5, 10, 20, 25, 50, 100. To reduce the work we quickly find that the smallest s such that $1978^s \equiv 1 \pmod 5$ is 4 and hence r must be a multiple of 4. That leaves 4, 20, 100 to examine.
We find $978^2 \equiv 109 \pmod{125}$, and hence $978^4 \equiv 6 \pmod{125}$. Hence $978^{20} \equiv 6^5 \equiv 36.91 \equiv 26 \pmod{125}$. So the smallest r is 100 and hence the solution to the problem is 3, 103.

**Problem 2**
**Solution** Suppose ABCD is a rectangle and X any point inside, then $XA^2 + XC^2 = XB^2 + XD^2$. This is most easily proved using coordinates. Take the origin O as the center of the rectangle and take OA to be the vector $\underline{a}$, and OB to be $\underline{b}$. Since it is a rectangle, $|\underline{a}| = |\underline{b}|$. Then OC is $-\underline{a}$ and OD is $-\underline{b}$. Let OX be $\underline{c}$. Then $XA^2 + XC^2 = (\underline{a} - \underline{c})^2 + (\underline{a} + \underline{c})^2 = 2\underline{a}^2 + 2\underline{c}^2 = 2\underline{b}^2 + 2\underline{c}^2 = XB^2 + XD^2$.
Let us fix U. Then the plane k perpendicular to PU through P cuts the sphere in a circle center C. V and W must lie on this circle. Take R so that PVRW is a rectangle. By the result just proved $CR^2 = 2CV^2 - CP^2$. OC is also perpendicular to the plane k. Extend it to X, so that CX = PU. Then extend XU to Y so that YR is perpendicular to k. Now $OY^2 = OX^2 + XY^2 = OX^2 + CR^2 = OX^2 + 2CV^2 - CP^2 = OU^2 - UX^2 + 2CV^2 - CP^2 = OU^2 - CP^2 + 2(OV^2 - OC^2) - CP^2 = 3OU^2 - 2OP^2$. Thus the locus of Y is a sphere.

**Problem 3**
**Solution** Let F = {f(1), f(2), f(3), ... }, G = {g(1), g(2), g(3), ... }, $N_n$ = {1, 2, 3, ... , n}. f(1) >= 1, so f(f(1)) >= 1 and hence g(1) >= 2. So 1 is not in G, and hence must be in F. It must be the smallest element of F and so f(1) = 1. Hence g(1) = 2. We can never have two successive integers n and n+1 in G, because if g(m) = n+1, then f(something) = n and so n is in F and G. Contradiction. In particular, 3 must be in F, and so f(2) = 3.
Suppose f(n) = k. Then g(n) = f(k) + 1. So $|N_{f(k)+1} \cap G| = n$. But $|N_{f(k)+1} \cap F| = k$, so n + k = f(k) + 1, or f(k) = n + k - 1. Hence g(n) = n + k. So n + k + 1 must be in F and hence f(k+1) = n + k + 1. This so given the value of f for n we can find it for k and k+1.
Using k+1 each time, we get, successively, f(2) = 3, f(4) = 6, f(7) = 11, f(12) = 19, f(20) = 32, f(33) = 53, f(54) = 87, f(88) = 142, f(143) = 231, f(232) = 375, which is not much help. Trying again with k, we get: f(3) = 4, f(4) = 6, f(6) = 9, f(9) = 14, f(14) = 22, f(22) = 35, f(35) = 56, f(56) = 90, f(90) = 145, f(145) = 234. Still not right, but we can try backing up slightly and using k+1: f(146) = 236. Still not right, we need to back up further: f(91) = 147, f(148) = 239, f(240) = 388.

**Problem 4**
**Solution** It is not a good idea to get bogged down in complicated formulae for the various radii. The solution is actually simple.
By symmetry the midpoint, M, is already on the angle bisector of A, so it is sufficient to show it is on the angle bisector of B. Let the angle bisector of A meet the circumcircle again at R. AP is a tangent to the circle touching AB at P, so angle PRQ = angle APQ =





angle ABC. Now the quadrilateral PBRM is cyclic because the angles PBR, PMR are both 90. Hence angle PBM = angle PRM = (angle PRQ)/2, so BM does indeed bisect angle B as claimed.

**Problem 5**
**Solution** We use the general rearrangement result: given $b_1 >= b_2 >= ... >= b_n$, and $c_1 <= c_2 <= ... <= c_n$, if $\{a_i\}$ is a permutation of $\{c_i\}$, then the sum from 1 to n of $a_ib_i$ is greater than or equal to the sum from 1 to n of $c_ib_i$. To prove it, suppose that i < j, but $a_i > a_j$. Then interchanging $a_i$ and $a_j$ does not increase the sum, because $(a_i - a_j)(b_i - b_j) >= 0$, and hence $a_ib_i + a_jb_j >= a_jb_i + a_ib_j$. By a series of such interchanges we transform $\{a_i\}$ into $\{c_i\}$ (for example, first swap $c_1$ into first place, then $c_2$ into second place and so on). Hence we do not increase the sum by permuting $\{a_i\}$ so that it is in increasing order. But now we have $a_i > i$, so we do not increase the sum by replacing $a_i$ by i and that gives the sum from 1 to n of 1/k.

**Problem 6**
**Solution** The trick is to use differences.
At least 6.329 = 1974, so at least 330 members come from the same country, call it C1. Let their numbers be $a_1 < a_2 < ... < a_{330}$. Now take the 329 differences $a_2 - a_1, a_3 - a_1, ... , a_{330} - a_1$. If any of them are in C1, then we are home, so suppose they are all in the other five countries.
At least 66 must come from the same country, call it C2. Write the 66 as $b_1 < b_2 < ... < b_{66}$. Now form the 65 differences $b_2 - b_1, b_3 - b_1, ... , b_{66} - b_1$. If any of them are in C2, then we are home. But each difference equals the difference of two of the original $a_i$s, so if it is in C1 we are also home.
So suppose they are all in the other four countries. At least 17 must come from the same country, call it C3. Write the 17 as $c_1 < c_2 < ... < c_{17}$. Now form the 16 differences $c_2 - c_1, c_3 - c_1, ... , c_{17} - c_1$. If any of them are in C3, we are home. Each difference equals the difference of two $b_i$s, so if any of them are in C2 we are home. [For example, consider $c_i - c_1$. Suppose $c_i = b_n - b_1$ and $c_1 = b_m - b_1$, then $c_i - c_1 = b_n - b_m$, as claimed.]. Each difference also equals the difference of two $a_i$s, so if any of them are in C1, we are also home. [For example, consider $c_i - c_1$, as before. Suppose $b_n = a_j - a_1$, $b_m = a_k - a_1$, then $c_i - c_1 = b_n - b_m = a_j - a_k$, as claimed.]
So suppose they are all in the other three countries. At least 6 must come from the same country, call it C4. We look at the 5 differences and conclude in the same way that at least 3 must come from C5. Now the 2 differences must both be in C6 and their difference must be in one of the C1, ... , C6 giving us the required sum.

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## 1979

**Problem 1**
**Solution** This is difficult.
The obvious step of combining adjacent terms to give 1/(n(n+1) is unhelpful. The trick is to separate out the negative terms:
1 - 1/2 + 1/3 - 1/4 + ... - 1/1318 + 1/1319 = 1 + 1/2 + 1/3 + ... + 1/1319 - 2(1/2 + 1/4 + ... + 1/1318) = 1/660 + 1/661 + ... + 1/1319.
and to notice that 660 + 1319 = 1979. Combine terms in pairs from the outside:
1/660 + 1/1319 = 1979/(660.1319); 1/661 + 1/1318 = 1979/(661.1318) etc.

There are an even number of terms, so this gives us a sum of terms 1979/m with m not divisible by 1979 (since 1979 is prime and so does not divide any product of smaller numbers). Hence the sum of the 1/m gives a rational number with denominator not divisible by 1979 and we are done.

**Problem 2**
**Solution** We show first that the $A_i$ are all the same color. If not then, there is a vertex, call it $A_1$, with edges $A_1A_2$, $A_1A_5$ of opposite color. Now consider the five edges $A_1B_i$. At least three of them must be the same color. Suppose it is green and that $A_1A_2$ is also green. Take the three edges to be $A_1B_i$, $A_1B_j$, $A_1B_k$. Then considering the triangles $A_1A_2B_i$, $A_1A_2B_j$, $A_1A_2B_k$, the three edges $A_2B_i$, $A_2B_j$, $A_2B_k$ must all be red. Two of $B_i$, $B_j$, $B_k$ must be adjacent, but if the resulting edge is red then we have an all red triangle with $A_2$, whilst if it is green we have an all green triangle with $A_1$. Contradiction. So the $A_i$ are all the same color. Similarly, the $B_i$ are all the same color. It remains to show that they are the same color. Suppose otherwise, so that the $A_i$ are green and the $B_i$ are red. Now we argue as before that 3 of the 5 edges $A_1B_i$ must be the same color. If it is red, then as before 2 of the 3 $B_i$ must be adjacent and that gives an all red triangle with $A_1$. So 3 of the 5 edges $A_1B_i$ must be green. Similarly, 3 of the 5 edges $A_2B_i$ must be green. But there must be a $B_i$ featuring in both sets and it forms an all green triangle with $A_1$ and $A_2$. Contradiction. So the $A_i$ and the $B_i$ are all the same color.

**Problem 3**
**Solution** Let the circles have centers C, D and let their other point of intersection be B. Take E on the circle center C and F on the circle center D so that EF is perpendicular to AB and meets it at A. Let the point on the circle center C be at P and the point on the other circle be at Q. Then angle ABP = half angle ACP = half angle ADQ = angle AFQ. Hence angle ABQ = 180 - angle AFQ = 180 - angle ABP, so PQ passes through B. Also angle BPE = 180 - angle BAE = 90 and angle BQE = 180 - angle BAF = 90. In other words the perpendiculars to PQ at its endpoints meet EF at its endpoints. Hence the perpendicular to PQ at its midpoint meets EF at its midpoint. So P and Q are equidistant from the midpoint of EF (which remains fixed as P and Q move).

**Problem 4**
**Solution** Consider the points R on a circle center P. Let X be the foot of the perpendicular from Q to k. Assume P is distinct from X, then we minimise QR (and hence maximise (QP + PR)/QR) for points R on the circle by taking R on the line PX. Moreover, R must lie on the same side of P as X. Hence if we allow R to vary over k, the points maximising (QP + PR)/QR must lie on the ray PX. Take S on the line PX on the opposite side of P from X so that PS = PQ. Then for points R on the ray PX we have (QP + PR)/QR = SR/QR = sin RQS/sin QSR. But sin QSR is fixed for points on the ray, so we maximise the ratio by taking angle RQS = 90. Thus there is a single point maximising the ratio.
If P = X, then we still require angle RQS = 90, but R is no longer restricted to a line, so it can be anywhere on a circle center P.

**Problem 5**
**Solution** Take $a^2$ x 1st equ - 2a x 2nd equ + 3rd equ. The rhs is 0. On the lhs the coefficient of $x_n$ is $a^2n - 2an^3 + n^5 = n(a - n^2)^2$. So the lhs is a sum of non-negative terms. Hence each term must be zero separately, so for each n either $x_n = 0$ or $a = n^2$. So there





are just 5 solutions, corresponding to a = 1, 4, 9, 16, 25. We can check that each of these gives a solution. [For $a = n^2$, $x_n = n$ and the other $x_i$ are zero.]

**Problem 6**
**Solution** Each jump changes the parity of the shortest distance to E. The parity is initially even, so an odd number of jumps cannot end at E. Hence $a_{2n-1} = 0$.
We derive a recurrence relation for $a_{2n}$. This is not easy to do directly, so we introduce $b_n$ which is the number of paths length n from C to E. Then we have immediately:
$a_{2n} = 2a_{2n-2} + 2b_{2n-2}$ for n > 1
$b_{2n} = 2b_{2n-2} + a_{2n-2}$ for n > 1
Hence, using the first equation: $a_{2n} - 2a_{2n-2} = 2a_{2n-2} - 4a_{2n-4} + 2b_{2n-2} - 4b_{2n-4}$ for n > 2.
Using the second equation, this leads to: $a_{2n} = 4a_{2n-2} - 2a_{2n-4}$ for n > 2. This is a linear recurrence relation with the general solution: $a_{2n} = a(2 + \sqrt{2})^{n-1} + b(2 - \sqrt{2})^{n-1}$. But we easily see directly that $a_4 = 2$, $a_6 = 8$ and we can now solve for the coefficients to get the solution given.

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## 1981

**Problem 1**
**Solution** We have PD.BC + PE.CA + PF.AB = 2 area of triangle. Now use Cauchy's inequality with $x_1 = \sqrt{(PD.BC)}$, $x_2 = \sqrt{(PE.CA)}$, $x_3 = \sqrt{(PF.AB)}$, and $y_1 = \sqrt{(BC/PD)}$, $y_2 = \sqrt{(CA/PE)}$, $y_3 = \sqrt{(AB/PF)}$. We get that $(BC + CA + AB)^2 < 2$ x area of triangle x (BC/PD + CA/PE + AB/PF) with equality only if $x_i/y_i$ = const, ie PD = PE = PF. So the unique minimum position for P is the incenter.

**Problem 2**
**Solution** Denote the binomial coefficient n!/(r!(n-r)!) by nCr.
Evidently nCr F(n,r) = 1 (n-1)C(r-1) + 2 (n-2)C(r-1) + ... + (n-r+1) (r-1)C(r-1). [The first term denotes the contribution from subsets with smallest element 1, the second term smallest element 2 and so on.]
Let the rhs be g(n,r). Then, using the relation (n-i)C(r-1) - (n-i-1)C(r-2) = (n-i-1)C(r-1), we find that g(n,r) - g(n-1,r-1) = g(n-1,r), and we can extend this relation to r=1 by taking g(n,0) = n+1 = (n+1)C1. But g(n,1) = 1 + 2 + ... + n = n(n+1)/2 = (n+1)C2. So it now follows by an easy induction that g(n,r) = (n+1)C(r+1) = nCr (n+1)/(r+1). Hence F(n,r) = (n+1)/(r+1).

**Problem 3**
**Solution** Experimenting with small values suggests that the solutions of $n^2 - mn - m^2$ = 1 or -1 are successive Fibonacci numbers. So suppose n > m is a solution. This suggests trying m+n, n: $(m+n)^2 - (m+n)n - n^2 = m^2 + mn - n^2 = -(n^2 - mn - m^2)$ = 1 or -1. So if n > m is a solution, then m+n, n is another solution. Running this forward from 2,1 gives 3,2; 5,3; 8,5; 13,8; 21,13; 34,21; 55,34; 89,55; 144,89; 233,144; 377,233; 610,377; 987,610; 1597,987; 2584,1597.
But how do we know that there are no other solutions? The trick is to run the recurrence the other way. For suppose n > m is a solution, then try m, n-m: $m^2 - m(n-m) - (n-m)^2 = m^2 + mn - n^2 = -(n^2 - mn - m^2)$ = 1 or -1, so that also satisfies the equation. Also if m > 1, then m > n-m (for if not, then n >= 2m, so n(n - m) >= $2m^2$, so $n^2 - nm - m^2 >= m^2$ > 1). So given a solution n > m with m > 1, we have a smaller solution m > n-m. This process must eventually terminate, so it must finish at a solution n, 1 with n > 1. But the

only such solution is 2, 1. Hence the starting solution must have been in the forward sequence from 2, 1.
Hence the solution to the problem stated is $1597^2 + 987^2$.

**Problem 4**
**Solution** (a) n = 3 is not possible. For suppose x was the largest number in the set. Then x cannot be divisible by 3 or any larger prime, so it must be a power of 2. But it cannot be a power of 2, because $2^m - 1$ is odd and $2^m - 2$ is not a positive integer divisible by $2^m$.
For k >= 2, the set 2k-1, 2k , ... , 4k-2 gives n = 2k. For k >= 3, so does the set 2k-5, 2k-4, ... , 4k-6. For k >= 2, the set 2k-2, 2k-3, ... , 4k-2 gives n = 2k+1. For k >= 4 so does the set 2k-6,2k-5, ... , 4k-6. So we have at least one set for every n >= 4, which answers (a).
(b) We also have at least two sets for every n >= 4 except possibly n = 4, 5, 7. For 5 we may take as a second set: 8, 9, 10, 11, 12, and for 7 we may take 6, 7, 8, 9 ,10, 11, 12. That leaves n = 4. Suppose x is the largest number in a set with n =4. x cannot be divisible by 5 or any larger prime, because x-1, x-2, x-3 will not be. Moreover, x cannot be divisible by 4, because then x-1 and x-3 will be odd, and x-2 only divisible by 2 (not 4). Similarly, it cannot be divisible by 9. So the only possibilities are 1, 2, 3, 6. But we also require x >= 4, which eliminates the first three. So the only solution for n = 4 is the one we have already found: 3, 4, 5, 6.

**Problem 5**
**Solution** Let the triangle be ABC. Let the center of the circle touching AB and AC be D, the center of the circle touching AB and BC be E, and the center of the circle touching AC and BC be F. Because the circles center D and E have the same radius the perpendiculars from D and E to AB have the same length, so DE is parallel to AB. Similarly EF is parallel to BC and FD is parallel to CA. Hence DEF is similar and similarly oriented to ABC. Moreover D must lie on the angle bisector of A since the circle center D touches AB and AC. Similarly E lies on the angle bisector of B and F lies on the angle bisector of C. Hence the incenter I of ABC is also the incenter of DEF and acts as a center of symmetry so that corresponding points P of ABC and P' of DEF lie on a line through I with PI/P'I having a fixed ratio. But OD = OE = OF since the three circles have equal radii, so O is the circumcenter of DEF. Hence it lies on a line with I and the circumcenter of ABC.

**Problem 6**
**Solution** f(1,n) = f(0,f(1,n-1)) = 1 + f(1,n-1). So f(1,n) = n + f(1,0) = n + f(0,1) = n + 2.
f(2,n) = f(1,f(2,n-1)) = f(2,n-1) + 2. So f(2,n) = 2n + f(2,0) = 2n + f(1,1) = 2n + 3.
f(3,n) = f(2,f(3,n-1)) = 2f(3,n-1) + 3. Let $u_n$ = f(3,n) + 3, then $u_n = 2u_{n-1}$. Also $u_0$ = f(3,0) + 3 = f(2,1) + 3 = 8. So $u_n = 2^{n+3}$, and f(3,n) = $2^{n+3} - 3$.
f(4,n) = f(3,f(4,n-1)) = $2^{f(4,n-1)+3} - 3$. f(4,0) = f(3,1) = $2^4 - 3$ = 13. We calculate two more terms to see the pattern: f(4,1) = $2^{24} - 3$, f(4,2) = $2^{224} - 3$. In fact it looks neater if we replace 4 by $2^2$, so that f(4,n) is a tower of n+3 2s less 3.

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## 1982

**Problem 1**
**Solution** We show that f(n) = [n/3] for n <= 9999, where [ ] denotes the integral part.





We show first that f(3) = 1. f(1) must be 0, otherwise f(2) - f(1) - f(1) would be negative. Hence f(3) = f(2) + f(1) + 0 or 1 = 0 or 1. But we are told f(3) > 0, so f(3) = 1. It follows by induction that f(3n) >= n. For f(3n+3) = f(3) + f(3n) + 0 or 1 = f(3n) + 1 or 2. Moreover if we ever get f(3n) > n, then the same argument shows that f(3m) > m for all m > n. But f(3.3333) = 3333, so f(3n) = n for all n <= 3333.
Now f(3n+1) = f(3n) + f(1) + 0 or 1 = n or n + 1. But 3n+1 = f(9n+3) >= f(6n+2) + f(3n+1) >= 3f(3n+1), so f(3n+1) < n+1. Hence f(3n+1) = n. Similarly, f(3n+2) = n. In particular f(1982) = 660.

**Problem 2**
**Solution** Let $B_i$ be the point of intersection of the interior angle bisector of the angle at $A_i$ with the opposite side. The first step is to figure out which side of $B_i T_i$ lies. Let $A_1$ be the largest angle, followed by $A_2$. Then $T_2$ lies between $A_1$ and $B_2$, $T_3$ lies between $A_1$ and $B_3$, and $T_1$ lies between $A_2$ and $B_1$. For angle $OB_2A_1 = 180 - A_1 - A_2/2 = A_3 + A_2/2$. But $A_3 + A_2/2 < A_1 + A_2/2$ and their sum is 180, so $A_3 + A_2/2 < 90$. Hence $T_2$ lies between $A_1$ and $B_2$. Similarly for the others.
Let O be the center of the incircle. Then angle $T_1OS_2$ = angle $T_1OT_2$ - 2 angle $T_2OB_2$ = $180 - A_3 - 2(90 - \text{angle } OB_2T_2) = 2(A_3 + A_2/2) - A_3 = A_2 + A_3$. A similar argument shows angle $T_1OS_3 = A_2 + A_3$. Hence $S_2S_3$ is parallel to $A_2A_3$.
Now angle $T_3OS_2$ = 360 - angle $T_3OT_1$ - angle $T_1OS_2$ = $360 - (180 - A_2) - (A_2 + A_3) = 180 - A_3 = A_1 + A_2$. Angle $T_3OS_1$ = angle $T_3OT_1$ + 2 angle $T_1OB_1$ = $(180 - A_2) + 2(90 - \text{angle } OB_1T_1) = 360 - A_2 - 2(A_3 + A_1/2) = 2(A_1 + A_2 + A_3) - A_2 - 2A_3 - A_1 = A_1 + A_2$ = angle $T_3OS_2$. So $S_1S_2$ is parallel to $A_1A_2$. Similarly we can show that $S_1S_3$ is parallel to $A_1A_3$.
So $S_1S_2S_3$ is similar to $A_1A_2A_3$ and turned through 180. But $M_1M_2M_3$ is also similar to $A_1A_2A_3$ and turned through 180. So $S_1S_2S_3$ and $M_1M_2M_3$ are similar and similarly oriented. Hence the lines through corresponding vertices are concurrent.

**Problem 3**
**Solution** (a) It is sufficient to show that the sum of the (infinite) sequence is at least 4. Let k be the greatest lower bound of the limits of all such sequences. Clearly k >= 1. Given any $\varepsilon > 0$, we can find a sequence $\{x_n\}$ with sum less than $k + \varepsilon$. But we may write the sum as:
$x_0^2/x_1 + x_1( (x_1/x_1)^2/(x_2/x_1) + (x_2/x_1)^2/(x_3/x_1) + ... + (x_n/x_1)^2/(x_{n+1}/x_1) + ... )$.
The term in brackets is another sum of the same type, so it is at least k. Hence $k + \varepsilon > 1/x_1 + x_1k$. This holds for all $\varepsilon > 0$, and so $k >= 1/x_1 + x_1k$. But $1/x_1 + x_1k >= 2\sqrt{k}$, so k >= 4.
(b) Let $x_n = 1/2^n$. Then $x_0^2/x_1 + x_1^2/x_2 + ... + x_{n-1}^2/x_n = 2 + 1 + 1/2 + ... + 1/2^{n-2} = 4 - 1/2^{n-2} < 4$.

**Problem 4**
**Solution** If x, y is a solution then so is y-x, -x. Hence also -y, x-y. If the first two are the same, then y = -x, and x = y-x = -2x, so x = y = 0, which is impossible, since n > 0. Similarly, if any other pair are the same.
$2891 \equiv 2 \pmod 9$ and there is no solution to $x^3 - 3xy^2 + y^3 \equiv 2 \pmod 9$. The two cubes are each $\equiv$ -1, 0 or 1, and the other term is $\equiv$ 0, 3 or 6, so the only solution is to have the cubes congruent to 1 and -1 and the other term congruent to 0. But the other term cannot be congruent to 0, unless one of x, y is a multiple of 3, in which case its cube is congruent to 0, not 1 or -1.

**Problem 5**
**Solution** For an inelegant solution one can use coordinates. The advantage of this type of approach is that it is quick and guaranteed to work! Take A as $(0,\sqrt{3})$, B as $(1,\sqrt{3})$, C as $(3/2,\sqrt{3}/2$, D as $(1,0)$. Take the point X, coordinates $(x,0)$, on ED. We find where the line BX cuts AC and CE. The general point on BX is $(k + (1-k)x, k\sqrt{3})$. If this is also the point M with AM/AC = r then we have: $k + (1-k)x = 3r/2$, $k\sqrt{3} = (1-r)\sqrt{3} + r\sqrt{3}/2$. Hence $k = 1 - r/2$, $r = 2/(4-x)$. Similarly, if it is the point N with CN/CE = r, then $k + (1-k)x = 3(1-r)/2$, $k\sqrt{3} = (1-r)\sqrt{3}/2$. Hence $k = (1-r)/2$ and $r = (2-x)/(2+x)$. Hence for the ratios to be equal we require $2/(4-x) = (2-x)/(2+x)$, so $x^2 - 8x + 4 = 0$. We also have $x < 1$, so $x = 4 - \sqrt{12}$. This gives $r = 1/\sqrt{3}$.
A more elegant solution uses the ratio theorem for the triangle EBC. We have CM/MX XB/BE EN/NC = -1. Hence $(1-r)/(r - 1/2)$ $(-1/4)$ $(1-r)/r = -1$. So $r = 1/\sqrt{3}$.

**Problem 6**
**Solution** Let the square be A'B'C'D'. The idea is to find points of L close to a particular point of A'D' but either side of an excursion to B'.
We say L approaches a point P' on the boundary of the square if there is a point P on L with PP' <= 1/2. We say L approaches P' before Q' if there is a point P on L which is nearer to $A_0$ (the starting point of L) than any point Q with QQ' <= 1/2.
Let A' be the first vertex of the square approached by L. L must subsequently approach both B' and D'. Suppose it approaches B' first. Let B be the first point on L with BB' <= 1/2. We can now divide L into two parts $L_1$, the path from $A_0$ to B, and $L_2$, the path from B to $A_n$.
Take X' to be the point on A'D' closest to D' which is approached by $L_1$. Let X be the corresponding point on $L_1$. Now every point on X'D' must be approached by $L_2$ (and X'D' is non-empty, because we know that D' is approached by L but not by $L_1$). So by compactness X' itself must be approached by $L_2$. Take Y to be the corresponding point on $L_2$. XY <= XX' + X'Y <= 1/2 + 1/2 = 1. Also BB' <= 1/2, so XB >= X'B' - XX' - BB' >= X'B' - 1 >= A'B' - 1 = 99. Similarly YB >= 99, so the path XY >= 198.

## 1983

**Problem 1**
**Solution** If $f(k) = 1$, then $f(x) = f(xf(k)) = kf(x)$, so $k =1$. Let $y = 1/f(x)$ and set $k = xf(y)$, then $f(k) = f(xf(y)) = yf(x) = 1$. Hence $f(1) = 1$ and $f(1/f(x)) = 1/x$. Also $f(f(y)) = f(1f(y)) = y$. Hence $f(1/x) = 1/f(x)$. Finally, let $z = f(y)$, so that $f(z) = y$. Then $f(xy) = f(xf(z)) = zf(x) = f(x)f(y)$.
Now notice that $f(xf(x)) = xf(x)$. Let $k = xf(x)$. We show that $k = 1$. $f(k^2) = f(k)f(k) = k^2$ and by a simple induction $f(k^n) = k^n$, so we cannot have $k > 1$, or $f(x)$ would not tend to 0 as x tends to infinity. But $f(1/k) = 1/k$ and the same argument shows that we cannot have $1/k > 1$. Hence $k = 1$.
So the only such function f is $f(x) = 1/x$.

**Problem 2**
**Solution** Let $P_1P_2$ and $O_1O_2$ meet at O. Let OA meet $C_2$ again at $A_2$. O is the center of similitude for $C_1$ and $C_2$ so angle $M_1AO_1$ = angle $M_2A_2O_2$. Hence if we can show that angle $M_2AO_2$ = angle $M_2A_2O_2$, then we are home.





Let X be the other point of intersection of the two circles. The key is to show that $A_2$, $M_2$ and X are collinear, for then angle $M_2AO_2$ = angle $M_2XO_2$ (by reflection) and $O_2A_2X$ is isosceles.
But since O is the center of similitude, $M_2A_2$ is parallel to $M_1A$, and by reflection angle $XM_2O$ = angle $AM_2O$, so we need to show that triangle $AM_1M_2$ is isosceles. Extend XA to meet $P_1P_2$ at Y. Then $YP_1^2 = YA.YX = YP_2^2$, so YX is the perpendicular bisector of $M_1M_2$, and hence $AM_1 = AM_2$ as required.

**Problem 3**
**Solution** We start with the lemma that bc - b - c is the largest number which cannot be written as mb + nc with m and n non-negative. [Proof: 0, c, 2c, ... , (b-1)c is a complete set of residues mod b. If r > (b-1)c - b, then r ≡ nc (mod b) for some 0 <= n <= b-1. But r > nc - b, so r = nc + mb for some m >= 0. That proves that every number larger than bc - b - c can be written as mb + nc with m and n non-negative. Now consider bc - b - c. It is ≡ (b-1)c (mod b), and not congruent to any nc with 0 <= n < b-1. So if bc - b - c = mb + nc, then n >= b-1. Hence mb + nc >= nc >= (b-1)c > bc - b - c. Contradiction.]
0, bc, 2bc, ... , (a-1)bc is a complete set of residues mod a. So given N > 2abc - ab - bc - ca we may take xbc ≡ N (mod a) with 0 <= x < a. But N - xbc > 2abc - ab - bc - ca - (a-1)bc = abc - ab - ca = a(bc - b - c). So N - xbc = ka, with k > bc - b - c. Hence we can find non-negative y, z so that k = zb + yc. Hence N = xbc + yca + zab.
Finally, we show that for N = 2abc - ab - bc - ca we cannot find non-negative x, y, z so that N = xbc + yca + zab. N ≡ -bc (mod a), so we must have x ≡ -1 (mod a) and hence x >= a-1. Similarly, y >= b-1, and z >= c-1. Hence xbc + yca + zab >= 3abc - ab - bc - ca > N. Contradiction.

**Problem 4**
**Solution** It does.
Suppose otherwise, that E is the disjoint union of e and e' with no right-angled triangles in either set. Take points X, Y, Z two-thirds of the way along BC, CA, AB respectively (so that BX/BC = 2/3 etc). Then two of X, Y, Z must be in the same set. Suppose X and Y are in e. Now YX is perpendicular to BC, so all points of BC apart from X must be in e'. Take W to be the foot of the perpendicular from Z to BC. Then B and W are in e', so Z must be in e. ZY is perpendicular to AC, so all points of AC apart from Y must be in e'. e' is now far too big. For example let M be the midpoint of BC, then AMC is in e' and right-angled.

**Problem 5**
**Solution** We may notice that an efficient way to build up a set with no APs length 3 is as follows. Having found $2^n$ numbers in $\{1, 2, ... , u_n\}$ we add the same pattern starting at $2u_n$, thus giving $2^{n+1}$ numbers in $\{1, 2, ... , 3u_n-1\}$. If x is in the first part and y, z in the second part, then 2y is at least $4u_n$, whereas x + z is less than $4u_n$, so x, y, z cannot be an AP length 3. If x and y are in the first part, and z in the second part, then 2y is at most $2u_n$, but x + z is more than $2u_n$, so x, y, z cannot be an AP length 3. To start the process off, we have the 4 numbers 1, 2, 4, 5 in {1, 2, 3, 4, 5}. So $u_2 = 5$. This gives $u_{11} = 88574$, in other words we can find 2048 numbers in the first 88574 with no AP length 3.
If we are lucky, we may notice that if we reduce each number in the set we have constructed by 1 we get the numbers which have no 2 when written base 3. This provides a neater approach. Take x, y, z with no 2 when written in base 3. Then 2y has

only 0s and 2s when written base 3. But x + z only has no 1s if x = z. So x, y, z cannot form an AP length 3. Also there are $2^{11}$ = 2048 numbers of this type with 11 digits or less and hence <= $11111111111_3$ = 88573.

**Problem 6**
**Solution** Put a = y + z, b = z + x, c = x + y. Then the triangle condition becomes simply x, y, z > 0. The inequality becomes (after some manipulation):
$xy^3 + yz^3 + zx^3 >= xyz(x + y + z)$.
Applying Cauchy's inequality we get $(xy^3 + yz^3 + zx^3)(z + x + y) >= xyz(y + z + x)^2$ with equality iff $xy^3/z = yz^3/x = zx^3/y$. So the inequality holds with equality iff x = y = z. Thus the original inequality holds with equality iff the triangle is equilateral.

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## 1984

**Problem 1**
**Solution** (1 - 2x)(1 - 2y)(1 - 2z) = 1 - 2(x + y + z) + 4(yz + zx + xy) - 8xyz = 4(yz + zx + xy) - 8xyz - 1. Hence yz + zx + xy - 2xyz = 1/4 (1 - 2x)(1 - 2y)(1 - 2z) + 1/4. By the arithmetic/geometric mean theorem (1 - 2x)(1 - 2y)(1 - 2z) <= $((1 - 2x + 1 - 2y + 1 - 2z)/3)^3$ = 1/27. So yz + zx + xy - 2xyz <= 1/4 28/27 = 7/27.

**Problem 2**
**Solution** We find that $(a + b)^7 - a^7 - b^7 = 7ab(a + b)(a^2 + ab + b^2)^2$. So we must find a, b such that $a^2 + ab + b^2$ is divisible by $7^3$.
At this point I found a = 18, b = 1 by trial and error.
A more systematic argument turns on noticing that $a^2 + ab + b^2 = (a^3 - b^3)/(a - b)$, so we are looking for a, b with $a^3 \equiv b^3 \pmod{7^3}$. We now remember that $a^{\phi(m)} \equiv 1 \pmod{m}$. But $\phi(7^3)$ = 2.3.49, so $a^3 \equiv 1$ (mod 343) if $a = n^{98}$. We find $2^{98} \equiv 18$ (343), which gives the solution 18, 1.
This approach does not give a flood of solutions. $n^{98} \equiv 0$, 1, 18, or 324. So the only solutions we get are 1, 18; 18, 324; 1, 324.

**Problem 3**
**Solution** Suppose the result is false. Let $C^1$ be any circle center O. Then the locus of points X such that $C(X) = C_1$ is a spiral from O to the point of intersection of OA and $C_1$. Every point of this spiral must be a different color from all points of the circle $C_1$. Hence every circle center O with radius smaller than $C_1$ must include a point of different color to those on $C_1$. Suppose there are n colors. Then by taking successively smaller circles $C_2, C_3, \dots , C_{n+1}$ we reach a contradiction, since each circle includes a point of different color to those on any of the larger circles.

**Problem 4**
**Solution** If AB and CD are parallel, then AB is tangent to the circle on diameter CD if and only if AB = CD and hence if and only if ABCD is a parallelogram. So the result is true.
Suppose then that AB and DC meet at O. Let M be the midpoint of AB and N the midpoint of CD. Let S be the foot of the perpendicular from N to AB, and T the foot of the perpendicular fromM to CD. We are given that MT = MA. OMT, ONS are similar, so OM/MT = ON/NS and hence OB/OA = (ON - NS)/(ON + NS). So AB is tangent to





the circle on diameter CD if and only if OB/OA = OC/OD which is the condition for BC to be parallel to AD.

**Problem 5**
**Solution** Given any diagonal AX, let B be the next vertex counterclockwise from A, and Y the next vertex counterclockwise from X. Then the diagonals AX and BY intersect at K. AK + KB > AB and XK + KY > XY, so AX + BY > AB + XY. Keeping A fixed and summing over X gives n - 3 cases. So if we then sum over A we get every diagonal appearing four times on the lhs and every side appearing 2(n-3) times on the rhs, giving 4d > 2(n-3)p.
Denote the vertices as $A_0, \dots , A_{n-1}$ and take subscripts mod n. The ends of a diagonal AX are connected by r sides and n-r sides. The idea of the upper limit is that its length is less than the sum of the shorter number of sides. Evaluating it is slightly awkward.
We consider n odd and n even separately. Let n = 2m+1. For the diagonal $A_iA_{i+r}$ with r <= m, we have $A_iA_{i+r} <= A_iA_{i+2} + \dots + A_iA_{i+r}$. Summing from r = 2 to m gives for the rhs $(m-1)A_iA_{i+1} + (m-1)A_{i+1}A_{i+2} + (m-2)A_{i+2}A_{i+3} + (m-3)A_{i+3}A_{i+4} + \dots + 1.A_{i+m-1}A_{i+m}$. Now summing over i gives d for the lhs and p( (m-1) + (1 + 2 + ... + m-1) ) = p( $(m^2$ + m - 2)/2 ) for the rhs. So we get 2d/p <= $m^2$ + m - 2 = [n/2] [(n+1)/2] - 2.
Let n = 2m. As before we have $A_iA_{i+r} <= A_iA_{i+2} + \dots + A_iA_{i+r}$ for 2 <= r <= m-1. We may also take $A_iA_{i+m}$ <= p/2. Summing as in the even case we get 2d/p = $m^2$ - 2 = [n/2] [(n+1)/2] - 2.

**Problem 6**
**Solution** a < c, so a(d - c) < c (d - c) and hence bc - ac < c(d - c). So b - a < d - c, or a + d > b + c, so k > m.
bc = ad, so $b(2^m - b) = a(2^k - a)$. Hence $b^2 - a^2 = 2^m(b - 2^{k-m}a)$. But $b^2 - a^2$ = (b + a)(b - a), and (b + a) and (b - a) cannot both be divisible by 4 (since a and b are odd), so $2^{m-1}$ must divide b + a or b - a. But if it divides b - a, then b - a >= $2^{m-1}$, so b and c > $2^{m-1}$ and b + c > $2^m$. Contradiction. Hence $2^{m-1}$ divides b + a. If b + a >= $2^m$ = b + c, then a >= c. Contradiction. Hence b + a = $2^{m-1}$.
So we have b = $2^{m-1}$ - a, c = $2^{m-1}$ + a, d = $2^k$ - a. Now using bc = ad gives: $2^ka = 2^{2m-2}$. But a is odd, so a = 1.

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## 1985

**Problem 1**
**Solution** Let the circle touch AD, CD, BC at L, M, N respectively. Take X on the line AD on the same side of A as D, so that AX = AO, where O is the center of the circle. Now the triangles OLX and OMC are congruent: OL = OM = radius of circle, angle OLX = angle OMC = 90, and angle OXL = 90 - A/2 = (180 - A)/2 = C/2 (since ABCD is cyclic) = angle OCM. Hence LX = MC. So OA = AL + MC. Similarly, OB = BN + MD. But MC = CN and MD = DL (tangents have equal length), so AB = OA + OB = AL + LD + CN + NB = AD + BC.

**Problem 2**
**Solution** n and k are relatively prime, so 0, k, 2k, ... , (n-1)k form a complete set of residues mod n. So k, 2k, ... , (n-1)k are congruent to the numbers 1, 2, ... , n-1 in some order. Suppose ik is congruent to r and (i+1)k is congruent to s. Then either s = r + k, or s = r + k - n. If s = r + k, then we have immediately that r = s - k and s have the same

color. If s = r + k - n, then r = n - (k - s), so r has the same color as k - s, and k - s has the same color as s. So in any case r and s have the same color. By giving i values from 1 to n-2 this establishes that all the numbers have the same color.

**Problem 3**

**Solution** If i is a power of 2, then all coefficients of $Q_i$ are even except the first and last. [There are various ways to prove this. Let iCr denote the rth coefficient, so iCr = i!/(r!(i-r)!). Suppose 0 < r < i. Then iCr = i-1Cr-1 i/r, but i-1Cr-1 is an integer and i is divisible by a higher power of 2 than r, hence iCr is even.]

Let $Q = Q_{i1} + ... + Q_{in}$. We use induction on $i_n$. If $i_n = 1$, then we must have n = 2, $i_1 = 0$, and $i_2 = 1$, so Q = 2 + x, which has the same number of odd coefficients as $Q_{i1} = 1$. So suppose it is true for smaller values of $i_n$. Take m a power of 2 so that m <= $i_n$ < 2m. We consider two cases $i_1$ >= m and $i_1$ < m.

Consider first $i_1$ >= m. Then $Q_{i1} = (1 + x)^m A$, $Q = (1 + x)^m B$, where A and B have degree less than m. Moreover, A and B are of the same form as $Q_{i1}$ and Q, (all the $i_j$s are reduced by m, so we have o(A) <= o(B) by induction. Also $o(Q_{i1}) = o((1 + x)^m A) = o(A + x^m A)$ = 2o(A) <= 2o(B) = $o(B + x^m B) = o((1 + x)^m B)$ = o(Q), which establishes the result for $i_n$.

It remains to consider the case $i_1$ < m. Take r so that $i_r$ < m, $i_{r+1}$ > m. Set $A = Q_{i1} + ... + Q_{ir}$, $(1 + x)^m B = Q_{ir+1} + ... + Q_{in}$, so that A and B have degree < m. Then $o(Q) = o(A + (1 + x)^m B) = o(A + B + x^m B)$ = o(A + B) + o(B). Now o(A - B) + o(B) >= o(A - B + B) = o(A), because a coefficient of A is only odd if just one of the corresponding coefficients of A - B and B is odd. But o(A - B) = o(A + B), because corresponding coefficients of A - B and A + B are either equal or of the same parity. Hence o(A + B) + o(B) >= o(A). But o(A) >= $o(Q_{ii})$ by induction. So we have established the result for $i_n$.

**Problem 4**

**Solution** Suppose we have a set of at least $3.2^n+1$ numbers whose prime divisors are all taken from a set of n. So each number can be written as $p_1^{r_1}...p_n^{r_n}$ for some non-negative integers $r_i$, where $p_i$ is the set of prime factors common to all the numbers. We classify each $r_i$ as even or odd. That gives $2^n$ possibilities. But there are more than $2^n + 1$ numbers, so two numbers have the same classification and hence their product is a square. Remove those two and look at the remaining numbers. There are still more than $2^n + 1$, so we can find another pair. We may repeat to find $2^n + 1$ pairs with a square product. [After removing $2^n$ pairs, there are still $2^n + 1$ numbers left, which is just enough to find the final pair.] But we may now classify these pairs according to whether each exponent in the square root of their product is odd or even. We must find two pairs with the same classification. The product of these four numbers is now a fourth power.

Applying this to the case given, there are 9 primes less than or equal to 23 (2, 3, 5, 7, 11, 13, 17, 19, 23), so we need at least 3.512 + 1 = 1537 numbers for the argument to work (and we have 1985).

The key is to find the 4th power in two stages, by first finding lots of squares. If we try to go directly to a 4th power, this type of argument does not work (we certainly need more than 5 numbers to be sure of finding four which sum to 0 mod 4, and $5^9$ is far too big).

**Problem 5**

**Solution** The three radical axes of the three circles taken in pairs, BM, NK and AC are concurrent. Let X be the point of intersection. [They cannot all be parallel or B and M





would coincide.] The first step is to show that XMNC is cyclic. The argument depends slightly on how the points are arranged. We may have: angle XMN = 180 - angle BMN = angle BKN = 180 - angle AKN = angle ACN = 180 - angle XCN, or we may have angle XMN = 180 - angle BMN = 180 - angle BKN = angle AKN = 180 - angle ACN = 180 - angle XCN.

Now XM.XB = XK.XN = $XO^2 - ON^2$. BM.BX = BN.BC = $BO^2 - ON^2$, so XM.XB - BM.BX = $XO^2 - BO^2$. But XM.XB - BM.BX = XB(XM - BM) = (XM + BM)(XM - BM) = $XM^2 - BM^2$. So $XO^2 - BO^2 = XM^2 - BM^2$. Hence OM is perpendicular to XB, or OMB is a right angle.

**Problem 6**

**Solution** Define $S_0(x) = x$, $S_n(x) = S_{n-1}(x) (S_{n-1}(x) + 1/n)$. The motivation for this is that $x_n = S_{n-1}(x_1)$.

$S_n(0) = 0$ and $S_n(1) > 1$ for all n > 1. Also $S_n(x)$ has non-negative coefficients, so it is strictly increasing in the range [0,1]. Hence we can find (unique) solutions $a_n$, $b_n$ to $S_n(a_n) = 1 - 1/n$, $S_n(b_n) = 1$.

$S_{n+1}(a_n) = S_n(a_n) (S_n(a_n) + 1/n) = 1 - 1/n > 1 - 1/(n+1)$, so $a_n < a_{n+1}$. Similarly, $S_{n+1}(b_n) = S_n(b_n) (S_n(b_n) + 1/n) = 1 + 1/n > 1$, so $b_n > b_{n+1}$. Thus $a_n$ is an increasing sequence and $b_n$ is a decreasing sequence with all $a_n$ less than all $b_n$. So we can certainly find at least one point $x_1$ which is greater than all the $a_n$ and less than all the $b_n$. Hence $1 - 1/n < S_n(x_1) < 1$ for all n. But $S_n(x_1) = x_{n+1}$. So $x_{n+1} < 1$ for all n. Also $x_n > 1 - 1/n$ implies that $x_{n+1} = x_n(x_n + 1/n) > x_n$. Finally, we obviously have $x_n > 0$. So the resulting series $x_n$ satisfies all the required conditions.

It remains to consider uniqueness. Suppose that there is an $x_1$ satisfying the conditions given. Then we must have $S_n(x_1)$ lying in the range 1 - 1/n, 1 for all n. [The lower limit follows from $x_{n+1} = x_n(x_n + 1/n)$.] Hence we must have $a_n < x_1 < b_n$ for all n. We show uniqueness by showing that $b_n - a_n$ tends to zero as n tends to infinity. Since all the coefficients of $S_n(x)$ are non-negative, it is has increasing derivative. $S_n(0) = 0$ and $S_n(b_n) = 1$, so for any x in the range 0, $b_n$ we have $S_n(x) <= x/b_n$. In particular, $1 - 1/n < a_n/b_n$. Hence $b_n - a_n <= b_n - b_n(1 - 1/n) = b_n/n < 1/n$, which tends to zero.

## 1986

**Problem 1**

**Solution** Consider residues mod 16. A perfect square must be 0, 1, 4 or 9 (mod 16). d must be 1, 5, 9, or 13 for 2d - 1 to have one of these values. However, if d is 1 or 13, then 13d - 1 is not one of these values. If d is 5 or 9, then 5d - 1 is not one of these values. So we cannot have all three of 2d - 1, 5d - 1, 13d - 1 perfect squares.

How does one find this solution? I thought the solution had to be either a congruence argument or a recurrence relation argument. For the latter $2d - 1 = r^2$ and $5d - 1 = s^2$ implies $2s^2 - 5r^2 = 3$. This appears to have a sparse, but infinite set of solutions. I thought one might find some type of recurrence relation between successive solutions and hence a formula for them. One could obtain a similar formula for the other equation ($2d - 1 = r^2$ and $13d - 1 = t^2$ implies $2t^2 - 13r^2 = 11$) and show that they had no common solution. However, despite some effort this led nowhere. So I went back to the congruence argument. I had already tried looking mod 4, 5, 6, 7, 8, 9. I now tried some higher numbers and hit on 16. In other words, I stumbled on the solution. Maybe there is a better way!

**Problem 2**
**Solution** The product of three successive rotations about the three vertices of a triangle must be a translation (see below). But that means that $P_{1986}$ (which is the result of 662 such operations, since 1986 = 3 x 662) can only be $P_0$ if it is the identity, for a translation by a non-zero amount would keep moving the point further away. It is now easy to show that it can only be the identity if the triangle is equilateral. Take a circle center $A_1$, radius $A_1A_2$ and take P on the circle so that a 120 degree clockwise rotation about $A_1$ brings P to $A_2$. Take a circle center $A_3$, radius $A_3A_2$ and take Q on the circle so that a 120 degree clockwise rotation about $A_3$ takes $A_2$ to Q. Then successive 120 degree clockwise rotations about $A_1$, $A_2$, $A_3$ take P to Q. So if these three are equivalent to the identity we must have P = Q. Hence angle $A_1A_2A_3$ = angle $A_1A_2P$ + angle $A_3A_2P$ = 30 + 30 = 60. Also $A_2P = 2A_1A_2\cos 30$ and $= 2A_2A_3\cos 30$. Hence $A_1A_2 = A_2A_3$. So $A_1A_2A_2$ is equilateral. Note in passing that it is not sufficient for the triangle to be equilateral. We also have to take the rotations in the right order. If we move around the vertices the opposite way, then we get a net translation.
It remains to show that the three rotations give a translation. Define rectangular coordinates (x, y) by taking $A_1$ to be the origin and $A_2$ to be (a, b). Let $A_3$ be (c, d). A clockwise rotation through 120 degrees about the origin takes (x, y) to (-x/2 + y√3/2, -x√3/2 - y/2). A clockwise rotation through 120 degrees about some other point (e, f) is obtained by subtracting (e, f) to get (x - e, y - f), the coordinates relative to (e, f), then rotating, then adding (e, f) to get the coordinates relative to (0, 0). Thus after the three rotations we will end up with a linear combination of x's, y's, a's, b's, c's and d's for each coordinate. But the linear combination of x's and y's must be just x for the x-coordinate and y for the y-coordinate, since three successive 120 degree rotations about the same point is the identity. Hence we end up with simply (x + constant, y + constant), in other words, a translation.
[Of course, there is nothing to stop you actually carrying out the computation. It makes things slightly easier to take the triangle to be (0, 0), (1, 0), (a, b). The net result turns out to be (x, y) goes to (x + 3a/2 - b√3/2, y - √3 + a√3/2 + 3b/2). For this to be the identity requires a = 1/2, b = √3/2. So the third vertex must make the triangle equilateral (and it must be on the correct side of the line joining the other two). This approach avoids the need for the argument in the first paragraph above, but is rather harder work.]

**Problem 3**
**Solution** Let S be the sum of the absolute value of each set of adjacent vertices, so if the integers are a, b, c, d, e, then S = |a| + |b| + |c| + |d| + |e| + |a + b| + |b + c| + |c + d| + |d + e| + |e + a| + |a + b + c| + |b + c + d| + |c + d + e| + |d + e + a| + |e + a + b| + |a + b + c + d| + |b + c + d + e| + |c + d + e + a| + |d + e + a + b| + |e + a + b + c| + |a + b + c + d + e|. Then the operation reduces S, but S is a greater than zero, so the process must terminate in a finite number of steps. So see that S is reduced, we can simply write out all the terms. Suppose the integers are a, b, c, d, e before the operation, and a+b, -b, b+c, d, e after it. We find that we mostly get the same terms before and after (although not in the same order), so that the sum S' after the operation is S - |a + c + d + e| + |a + 2b + c + d + e|. Certainly a + c + d + e > a + 2b + c + d + e since b is negative, and a + c + d + e > -(a + 2b+ c + d + e) because a + b + c + d + e > 0.

**Problem 4**
**Solution** Take AB = 2 and let M be the midpoint of AB. Take coordinates with origin at A, x-axis as AB and y-axis directed inside the n-gon. Let Z move along AB from B





towards A. Let the angle YZA be t. Let the coordinates of X be (x, y). Angle YZX = pi/2 - pi/n, so XZ = 1/sin pi/n and y = XZ sin(t + pi/2 - pi/n) = sin t + cot pi/n cos t. BY sin 2pi/n = YZ sin t = 2 sin t. MX = cot pi/n. So x = MY cos t - BY cos 2pi/n + MX sin t = cos t + (cot pi/n - 2 cot 2pi/n) sin t = cos t + tan pi/n sin t = y tan pi/n. Thus the locus of X is a star formed of n lines segments emanating from O. X moves out from O to the tip of a line segement and then back to O, then out along the next segment and so on. $x^2 + y^2 = (1/\sin^2 pi/n + 1/\cos^2 pi/n) \cos^2(t + pi/n)$. Thus the length of each segment is (1 - cos pi/n)/(sin pi/n cos pi/n).

**Problem 5**
**Solution** f(x+2) = f(xf(2)) f(2) = 0. So f(x) = 0 for all x >= 2.
f(y) f((2-y)f(y)) = f(2) = 0. So if y < 2, then f((2-y) f(y)) = 0 and hence (2 - y) f(y) >= 2, or f(y) >= 2/(2 - y).
Suppose that for some $y_0$ we have $f(y_0) > 2/(2 - y_0)$, then we can find $y_1 > y_0$ (and $y_1 < 2$) so that $f(y_0) = 2/(2 - y_1)$. Now let $x_1 = 2 - y_1$. Then $f(x_1f(y_0)) = f(2) = 0$, so $f(x_1 + y_0) = 0$. But $x_1 + y_0 < 2$. Contradiction. So we must have f(x) = 2/(2 - x) for all x < 2.
We have thus established that if a function f meets the conditions then it must be defined as above. It remains to prove that with this definition f does meet the conditions. Clearly f(2) = 0 and f(x) is non-zero for 0 <= x < 2. f(xf(y)) = f(2x/(2 - y)). If 2x/(2 - y) >= 2, then f(xf(y)) = 0. But it also follows that x + y >= 2, and so f(x + y) = 0 and hence f(xf(y)) f(y) = f(x + y) as required. If 2x/(2 - y) < 2, then f(xf(y)) f(y) = 2/(2 - 2x/(2-y)) 2/(2 - y) = 2/(2 - x - y) = f(x + y). So the unique function satisfying the conditions is:
f(x) = 0 for x >= 2, and 2/(2 - x) for 0 <= x < 2.

**Problem 6**
**Solution** Answer: yes.
We prove the result by induction on the number n of points. It is clearly true for n = 1. Suppose it is true for all numbers less than n. Pick an arbitary point P and color it red. Now take a point in the same row and color it white. Take a point in the same column as the new point and color it red. Continue until either you run out of eligible points or you pick a point in the same column as P. The process must terminate because there are only finitely many points. Suppose the last point picked is Q. Let S be the set of points picked.
If Q is in the same column as P, then it is colored white (because the "same row" points are all white, and the "same column" points are all red). Now every row and column contains an equal number of red points of S and of white points of S. By induction we can color the points excluding those in S, then the difference between the numbers of red and white points in each row and column will be unaffected by adding the points in S and so we will have a coloring for the whole set. This completes the induction for the case where Q is in the same column as P.
If it is not, then continue the path backwards from P. In other words, pick a point in the same column as P and color it white. Then pick a point in the same row as the new point and color it red and so on. Continue until either you run out of eligible points or you pick a point to pair with Q. If Q was picked as being in the same row as its predecessor, this means a point in the same column as Q; if Q was picked as being in the same column as its predecessor, this means a point in the same row as Q. Again the process must terminate. Suppose the last point picked is R. Let S be the set of all points picked. If R pairs with Q, then we can complete the coloring by induction as before. Suppose S does not pair with Q. Then there is a line (meaning a row or column) containing Q and no uncolored points. There is also a line containing R and no uncolored points. These

two lines have an excess of one red or one white. All other lines contain equal number of red and white points of S. Now color the points outside S by induction. This gives a coloring for the whole set, because no line with a color excess in S has any points outside S. So we have completed the induction.

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## 1987

**Problem 1**
**Solution** We show first that the number of permutations of n objects with no fixed points is $n!(1/0! - 1/1! + 1/2! - ... + (-1)^n/n!)$. This follows immediately from the law of inclusion and exclusion: let $N_i$ be the number which fix i, $N_{ij}$ the number which fix i and j, and so on. Then $N_0$, the number with no fixed points, is n! - all $N_i$ + all $N_{ij}$ - ... + $(-1)^n N_{1...n}$. But $N_i = (n-1)!$, $N_{ij} = (n-2)!$ and so on. So $N_0 = n! ( 1 - 1/1! + ... + (-1)^r(n-r)!/(r! (n-r)!) + ... + (-1)^n/n!) = n! (1/0! - 1/1! + ... + (-1)^n/n!)$.
Hence the number of permutations of n objects with exactly r fixed points = no. of ways of choosing the r fixed points x no. of perms of the remaining n - r points with no fixed points = $n!/(r! (n-r)!)$ x $(n-r)! (1/0! - 1/1! + ... + (-1)^{n-r}/(n-r)! )$. Thus we wish to prove that the sum from r = 1 to n of $1/(r-1)! (1/0! - 1/1! + ... + (-1)^{n-r}/(n-r)! )$ is 1. We use induction on n. It is true for n = 1. Suppose it is true for n. Then the sum for n+1 less the sum for n is: $1/0! (-1)^n/n! + 1/1! (-1)^{n-1}/(n-1)! + ... + 1/n! 1/0! = 1/n! (1 - 1)^n = 0$. Hence it is true for n + 1, and hence for all n.
*Comment*
This is a plodding solution. If you happen to know the result for no fixed points (which many people do), then it is essentially a routine induction.

**Second solution**
Count all pairs (x, s) where s is a permutation with x a fixed point of x. Clearly, if we fix x, then there are (n-1)! possible permutations s. So the total count is n!. But if we count the number of permutations s with exactly k fixed points, then we get the sum in the question.

**Problem 2**
**Solution** AKL and AML are congruent, so KM is perpendicular to AN and area AKNM = KM.AN/2.
AKLM is cyclic (2 opposite right angles), so angle AKM = angle ALM and hence KM/sin BAC = AM/sin AKM (sine rule) = AM/sin ALM = AL.
ABL and ANC are similar, so AB.AC = AN.AL. Hence area ABC = 1/2 AB.AC sin BAC = 1/2 AN.AL sin BAC = 1/2 AN.KM = area AKNM.

**Problem 3**
**Solution** This is an application of the pigeon-hole principle.
Assume first that all $x_i$ are non-negative. Observe that the sum of the $x_i$ is at most $\sqrt{n}$. [This is a well-known variant, $(\mathrm{Sum}_{1<=i<=n}\, x_i)^2 <= n\, \mathrm{Sum}_{1<=i<=n}\, x_i^2$, of the AM-GM result. See, for example, Arthur Engel, Problem Solving Strategies, Springer 1998, p163, ISBN 0387982191].
Consider the $k^n$ possible values of $\mathrm{Sum}_{1<=i<=n}\, b_i x_i$, where each $b_i$ is an integer in the range [0,k-1]. Each value must lie in the interval [0, k-1 $\sqrt{n}$]. Divide this into $k^n$-1 equal subintervals. Two values must lie in the same subinterval. Take their difference. Its





coefficients are the required $a_i$. Finally, if any $x_i$ are negative, solve for the absolute values and then flip signs in the $a_i$.

**Problem 4**
**Solution** We prove that if f(f(n)) = n + k for all n, where k is a fixed positive integer, then k must be even. If k = 2h, then we may take f(n) = n + h.
Suppose f(m) = n with m ≡ n (mod k). Then by an easy induction on r we find f(m + kr) = n + kr, f(n + kr) = m + k(r+1). We show this leads to a contradiction. Suppose m < n, so n = m + ks for some s > 0. Then f(n) = f(m + ks) = n + ks. But f(n) = m + k, so m = n + k(s - 1) >= n. Contradiction. So we must have m >= n, so m = n + ks for some s >= 0. But now f(m + k) = f(n + k(s+1)) = m + k(s + 2). But f(m + k) = n + k, so n = m + k(s + 1) > n. Contradiction.
So if f(m) = n, then m and n have different residues mod k. Suppose they have $r_1$ and $r_2$ respectively. Then the same induction shows that all sufficiently large s ≡ $r_1$ (mod k) have f(s) ≡ $r_2$ (mod k), and that all sufficiently large s ≡ $r_2$ (mod k) have f(s) ≡ $r_1$ (mod k). Hence if m has a different residue r mod k, then f(m) cannot have residue $r_1$ or $r_2$. For if f(m) had residue $r_1$, then the same argument would show that all sufficiently large numbers with residue $r_1$ had f(m) ≡ r (mod k). Thus the residues form pairs, so that if a number is congruent to a particular residue, then f of the number is congruent to the pair of the residue. But this is impossible for k odd.

**Problem 5**
**Solution** Let $x_n$ be the point with coordinates $(n, n^2)$ for n = 1, 2, 3, ... . We show that the distance between any two points is irrational and that the triangle determined by any 3 points has non-zero rational area.
Take n > m. $|x_n - x_m|$ is the hypoteneuse of a triangle with sides n - m and $n^2 - m^2 = (n - m)(n + m)$. So $|x_n - x_m| = (n - m)\sqrt{(1 + (n+m)^2)}$. Now $(n + m)^2 < (n + m)^2 + 1 < (n + m + 1)^2 = (n + m)^2 + 1 + 2(n + m)$, so $(n + m)^2 + 1$ is not a perfect square. Hence its square root is irrational. [For this we may use the classical argument. Let N' be a non-square and suppose $\sqrt{N'}$ is rational. Since N' is a non-square we must be able to find a prime p such that $p^{2a+1}$ divides N' but $p^{2a+2}$ does not divide N' for some a >= 0. Define $N = N'/p^{2a}$. Then $\sqrt{N} = (\sqrt{N'})/p^a$, which is also rational. So we have a prime p such that p divides N, but $p^2$ does not divide N. Take $\sqrt{N} = r/s$ with r and s relatively prime. So $s^2N = r^2$. Now p must divide r, hence $p^2$ divides $r^2$ and so p divides $s^2$. Hence p divides s. So r and s have a common factor. Contradiction. Hence non-squares have irrational square roots.]
Now take a < b < c. Let B be the point $(b, a^2)$, C the point $(c, a^2)$, and D the point $(c, b^2)$. Area $x_a x_b x_c$ = area $x_a x_c C$ - area $x_a x_b B$ - area $x_b x_c D$ - area $x_b DCB = (c - a)(c^2 - a^2)/2 - (b - a)(b^2 - a^2)/2 - (c - b)(c^2 - b^2)/2 - (c - b)(b^2 - a^2)$ which is rational.

**Problem 6**
**Solution** First observe that if m is relatively prime to b + 1, b + 2, ... , 2b - 1, 2b, then it is not divisible by any number less than 2b. For if c <= b, then take the largest j >= 0 such that $2^j c <= b$. Then $2^{j+1}c$ lies in the range b + 1, ... , 2b, so it is relatively prime to m. Hence c is also. If we also have that $(2b + 1)^2 > m$, then we can conclude that m must be prime, since if it were composite it would have a factor <= $\sqrt{m}$.
Let $n = 3r^2 + h$, where 0 <= h < 6r + 3, so that r is the greatest integer less than or equal to $\sqrt{(n/3)}$. We also take r >= 1. That excludes the value n = 2, but for n = 2, the result is vacuous, so nothing is lost.

Assume that $n + k(k+1)$ is prime for $k = 0, 1, ... , r$. We show by induction that $N = n + (r + s)(r + s + 1)$ is prime for $s = 1, 2, ... , n - r - 2$. By the observation above, it is sufficient to show that $(2r + 2s + 1)^2 > N$, and that N is relatively prime to all of $r + s + 1, r + s + 2, ... , 2r + 2s$. We have $(2r + 2s + 1)^2 = 4r^2 + 8rs + 4s^2 + 4r + 4s + 1$. Since r, s >= 1, we have $4s + 1 > s + 2$, $4s^2 > s^2$, and $6rs > 3r$. Hence $(2r + 2s + 1)^2 > 4r^2 + 2rs + s^2 + 7r + s + 2 = 3r^2 + 6r + 2 + (r + s)(r + s + 1) >= N$.
Now if N has a factor which divides $2r - i$ with i in the range $-2s$ to $r - s - 1$, then so does $N - (i + 2s + 1)(2r - i) = n + (r - i - s - 1)(r - i - s)$ which has the form $n + s'(s'+1)$ with s' in the range 0 to $r + s - 1$. But $n + s'(s' + 1)$ is prime by induction (or absolutely for s = 1), so the only way it can have a factor in common with $2r - i$ is if it divides $2r - i$. But $2r - i <= 2r + 2s <= 2n - 4 < 2n$ and $n + s'(s' + 1) >= n$, so if $n + s'(s' + 1)$ has a factor in common with $2r - i$, then it equals $2r - i = s + r + 1 + s'$. Hence $s'^2 = s - (n - r - 1) < 0$, which is not possible. So we can conclude that N is relatively prime to all of $r + s + 1, ... , 2r + 2s$ and hence prime.

## 1988

**Problem 1**
**Solution** (i) Let M be the midpoint of BC. Let PM = x. Let BC meet the small circle again at Q. Let O be the center of the circles. Since angle APQ = 90 degrees, AQ is a diameter of the small circle, so its length is 2r. Hence $AP^2 = 4r^2 - 4x^2$. $BM^2 = R^2 - OM^2 = R^2 - (r^2 - x^2)$. That is essentially all we need, because we now have: $AB^2 + AC^2 + BC^2 = (AP^2 + (BM - x)^2) + (AP^2 + (BM + x)^2) + 4BM^2 = 2AP^2 + 6BM^2 + 2x^2 = 2(4r^2 - 4x^2) + 6(R^2 - r^2 + x^2) + 2x^2 = 6R^2 + 2r^2$ , which is independent of x.
(ii) M is the midpoint of BC and PQ since the circles have a common center. If we shrink the small circle by a factor 2 with P as center, then Q moves to M, and hence the locus of M is the circle diameter OP.

**Problem 2**
**Solution** Answer: n even.
Each of the 2n elements of $A_i$ belongs to at least one other $A_j$ because of (iii). But given another $A_j$ it cannot contain more than one element of $A_i$ because of (ii). There are just 2n other $A_j$ available, so each must contain exactly one element of $A_i$. Hence we can strengthen (iii) to every element of B belongs to exactly two of the As.
This shows that the arrangement is essentially unique. We may call the element of B which belongs to $A_i$ and $A_j$ (i,j). Then $A_i$ contains the 2n elements (i, j) with j not i.
$|B| = 1/2$ x no. of As x size of each $A = n(2n+1)$. If the labeling with 0s and 1s is possible, then if we list all the elements in each A, $n(2n+1)$ out of the $2n(2n+1)$ elements have value 0. But each element appears twice in this list, so $n(2n+1)$ must be even. Hence n must be even.
We show by induction that a labeling is always possible for n even. If n = 2, there is certainly a labeling. For example, we may assign 0 to (1,2), (1,3), (2,4), (3,5), (4,5).
Now suppose we have a labeling for n. For n + 2, we label (i , j) 0 if it was labeled 0 for n or if it is:
(i, 2n+2) or (i, 2n+3) for i = 1, 2, ... , n+1
(i, 2n+4) or (i, 2n+5) for i = n+2, n+3, ... , 2n+1
(2n+2, 2n+4), (2n+3, 2n+5), (2n+4,2n+5).
For i = 1, 2, ... n+1, $A_i$ has n elements (i, j) labeled zero with j <= 2n+1 and also (i, 2n+2) and (i, 2n+3), giving n+2 in all. For i = n+2, n+3, ... , 2n+1, $A_i$ has n elements (i,





j) labeled zero with j <= 2n+1 and also (i, 2n+4) and (i, 2n+5), giving n+2 in all. $A_{2n+2}$ has the n+1 elements (i, 2n+2) with i <= n+1 and also (2n+2, 2n+4), giving n+2 in all. $A_{2n+3}$ has the n+1 elements (i, 2n+3) for i <= n+1 and also (2n+3, 2n+5), giving n+2 in all. $A_{2n+4}$ has the n elements (i, 2n+4) with n+2 <= i <= 2n+1 and also (2n+2, 2n+4) and (2n+4, 2n+5), giving n+2 in all. Finally $A_{2n+5}$ has the n elements (i, 2n+5) with n+2 <= i <= 2n+1 and also (2n+3, 2n+5) and (2n+4, 2n+5), giving n+2 in all.

**Problem 3**
**Solution** Answer: 92.
f(n) is always odd. If $n = b_{r+1}b_r...b_2b_1b_0$ in binary and n is odd, so that $b_{r+1} = b_0 = 1$, then $f(n) = b_{r+1}b_1b_2...b_rb_0$. If n has r+2 binary digits with r > 0, then there are $2^{[(r+1)/2]}$ numbers with the central r digits symmetrical, so that f(n) = n (because we can choose the central digit and those lying before it arbitarily, the rest are then determined). Also there is one number with 1 digit (1) and one number with two digits (3) satisfying f(n) = 1. So we find a total of 1 + 1 + 2 + 2 + 4 + 4 + 8 + 8 + 16 + 16 = 62 numbers in the range 1 to 1023 with f(n) = n. 1988 = 11111000011. So we also have all 32 numbers in the range 1023 to 2047 except for 11111111111 and 11111011111, giving another 30, or 92 in total.
It remains to prove the assertions above. f(n) odd follows by an easy induction. Next we show that if $2^m < 2n+1 < 2^{m+1}$, then $f(2n+1) = f(n) + 2^m$. Again we use induction. It is true for m = 1 (f(3) = f(1) + 2). So suppose it is true for 1, 2, ... , m. Take 4n+1 so that $2^{m+1} < 4n+1 < 2^{m+2}$, then $f(4n+1) = 2f(2n+1) - f(n) = 2(f(n) + 2^m) - f(n) = f(n) + 2^{m+1} = f(2n) + 2^{m+1}$, so it is true for 4n+1. Similarly, if 4n+3 satisfies, $2^{m+1} < 4n+3 < 2^{m+2}$, then $f(4n+3) = 3f(2n+1) - 2f(n) = f(2n+1) + 2(f(n) + 2^m) - 2f(n) = f(2n+1) + 2^{m+1}$, so it is true for 4n+3 and hence for m+1.
Finally, we prove the formula for f(2n+1). Let $2n+1 = b_{r+1}b_r...b_2b_1b_0$ with $b_0 = b_{r+1} = 1$. We use induction on r. So assume it is true for smaller values. Say $b_1 = ... = b_s = 0$ and $b_{s+1} = 1$ (we may have s = 0, so that we have simply $b_1 = 1$). Then $n = b_{r+1} ... b_1$ and $f(n) = b_{r+1}b_{s+2}b_{s+3}...b_rb_{s+1}$ by induction. So $f(n) + 2^{r+1} = b_{r+1}0...0b_{r+1}b_{s+2}...b_rb_{s+1}$, where there are s zeros. But we may write this as $b_{r+1}b_1...b_sb_{s+1}...b_rb_{r+1}$, since $b_1 = ... = b_s = 0$, and $b_{s+1} = b_{r+1} = 1$. But that is the formula for f(2n+1), so we have completed the induction.

**Problem 4**
**Solution** Let $f(x) = 1/(x - 1) + 2/(x - 2) + 3/(x - 3) + ... + 70/(x - 70)$. For any integer n, $n/(x - n)$ is strictly monotonically decreasing except at x = n, where it is discontinuous. Hence f(x) is strictly monotonically decreasing except at x = 1, 2, ... , 70. For n = any of 1, 2, ... , 70, $n/(x - n)$ tends to plus infinity as x tends to n from above, whilst the other terms $m/(x - m)$ remain bounded. Hence f(x) tends to plus infinity as x tends to n from above. Similarly, f(x) tends to minus infinity as x tends to n from below. Thus in each of the intervals (n, n+1) for n = 1, ... , 69, f(x) decreases monotonically from plus infinity to minus infinity and hence f(x) = 5/4 has a single foot $x_n$. Also f(x) >= 5/4 for x in (n, $x_n$] and f(x) < 5/4 for x in ($x_n$, n+1). If x < 0, then every term is negative and hence f(x) < 0 < 5/4. Finally, as x tends to infinity, every term tends to zero, so f(x) tends to zero. Hence f(x) decreases monotonically from plus infinity to zero over the range [70, infinity]. Hence f(x) = 5/4 has a single root $x_{70}$ in this range and f(x) >= 5/4 for x in (70, $x_{70}$] and f(x) < 5/4 for x > $x_{70}$. Thus we have established that f(x) >= 5/4 for x in any of the disjoint intervals (1, $x_1$], (2, $x_2$], ... , (70, $x_{70}$] and f(x) < 5/4 elsewhere.
The total length of these intervals is $(x_1 - 1) + ... + (x_{70} - 70) = (x_1 + ... + x_{70}) - (1 + ... + 70)$. The $x_i$ are the roots of the 70th order polynomial obtained from $1/(x - 1) + 2/(x - 2) + 3/(x - 3) + ... + 70/(x - 70) = 5/4$ by multiplying both sides by $(x - 1) ... (x - 70)$. The

sum of the roots is minus the coefficient of $x^{69}$ divided by the coefficient of $x^{70}$. The coefficient of $x^{70}$ is simply k, and the coefficient of $x^{69}$ is - (1 + 2 + ... + 70)k - (1 + ... + 70). Hence the sum of the roots is (1 + ... + 70)(1 + k)/k and the total length of the intervals is (1 + ... + 70)/k = 1/2 70.71 4/5 = 28.71 = 1988.

**Problem 5**
**Solution** The key is to show that AK = AL = AD. We do this indirectly. Take K' on AB and L' on AC so that AK' = AL' = AD. Let the perpendicular to AB at K' meet the line AD at X. Then the triangles AK'X and ADB are congruent. Let J be the incenter of ADB and let r be the in-radius of ADB. Then J lies on the angle bisector of angle BAD a distance r from the line AD. Hence it is also the incenter of AK'X. Hence JK' bisects the right angle AK'X, so angle AK'J = 45 degrees and so J lies on K'L'. An exactly similar argument shows that I, the incenter of ADC, also lies on K'L'. Hence we can identify K and K', and L and L'.
The area of AKL is AK.AL/2 = $AD^2/2$, and the area of ABC is BC.AD/2, so we wish to show that 2AD <= BC. Let M be the midpoint of BC. Then AM is the hypoteneuse of AMD, so AM >= AD with equality if and only if D = M. Hence 2AD <= 2AM = BC with equality if and only if AB = AC.

**Problem 6**
**Solution** A little experimentation reveals the following solutions: a, $a^3$ giving $a^2$; $a^3$, $a^5$ - a giving $a^2$; and the recursive $a_1 = 2$, $b_1 = 8$, $a_{n+1} = b_n$, $b_{n+1} = 4b_n - a_n$ giving 4. The latter may lead us to: if $a^2 + b^2 = k(ab + 1)$, then take A = b, B = kb - a, and then $A^2 + B^2 = k(AB + 1)$. Finally, we may notice that this can be used to go down as well as up.
So starting again suppose that a, b, k is a solution in positive integers to $a^2 + b^2 = k(ab + 1)$. If a = b, then $2a^2 = k(a^2 + 1)$. So $a^2$ must divide k. But that implies that a = b = k = 1. Let us assume we do not have this trivial solution, so we may take a < b. We also show that $a^3 > b$. For $(b/a - 1/a)(ab + 1) = b^2 + b/a - b - 1/a < b^2 < a^2 + b^2$. So k > b/a - 1/a. But if $a^3 < b$, then $b/a\ (ab + 1) > b^2 + a^2$, so k < b/a. But now b > ak and lt; ak + 1, which is impossible. It follows that k >= b/a.
Now define A = ka - b, B = a. Then we can easily verify that A, B, k also satisfies $a^2 + b^2 = k(ab + 1)$, and B and k are positive integers. Also a < b implies $a^2 + b^2 < ab + b^2 < ab + b^2 + 1 + b/a = (ab + 1)(1 + b/a)$, and hence k < 1 + b/a, so ka - b < a. Finally, since k > b/a, ka - b >= 0. If ka - b > 0, then we have another smaller solution, in which case we can repeat the process. But we cannot have an infinite sequence of decreasing numbers all greater than zero, so we must eventually get A = ka - b = 0. But now $A^2 - B^2 = k(AB + 1)$, so $k = B^2$. k was unchanged during the descent, so k is a perfect square.

## 1989

**Problem 1**
**Solution** We construct 116 sets of three numbers. Each set sums to 3 x 995 = 2985. The 348 numbers involved form 174 pairs {r, 1990 - r}. At this point we are essentially done. We take a 117th set which has one {r, 1990 - r} pair and 995. The original 1989 numbers comprise 995 and 994 {r, 1990 - r} pairs. We have used up 995 and 175 pairs, leaving just 819 pairs. We now add 7 pairs to each of our 117 sets, bringing the total of each set up to 2985 + 7.1990 = 1990 x 17/2.
It remains to exhibit the 116 sets. There are many possibilities. We start with:
301, 801, 1883 and the "complementary" set 1990 - 301 = 1689, 1990 - 801 = 1189,





1990 - 1883 = 107. We then add one to each of the first two numbers to get:
302, 802, 1881 and 1688, 1188, 109, and so on:
303, 803, 1879 and 1687, 1187, 111,
...
358, 858, 1769 and 1632, 1132, 221.
We can immediately see that these triples are all disjoint. So the construction is complete.

**Problem 2**
**Solution** Let I be the point of intersection of $AA_0$, $BB_0$, $CC_0$ (the in-center). BIC = 180 - 1/2 ABC - 1/2 BCA = 180 - 1/2 (180 - CAB) = 90 + 1/2 CAB. Hence $CA_1B$ = 180 - CAB [$BA_1CA$ is cyclic] = 2(180 - BIC) = $2CA_0B$. But $A_1B = A_1C$, so $A_1$ is the center of the circumcircle of $BCA_0$. But I lies on this circumcircle ($IBA_0 = ICA_0 = 90$), and hence $A_1A_0 = A_1I$.
Hence area $IBA_1$ = area $A_0BA_1$ and area $ICA_1$ = area $A_0CA_1$. Hence area $IBA_0C$ = 2 area $IBA_1C$. Similarly, area $ICB_0A$ = 2 area $ICB_1A$ and area $IAC_0B$ = 2 area $IAC_1B$. Hence area $A_0B_0C_0$ = 2 area hexagon $AB_1CA_1BC_1$.
Let H be the orthocentre of ABC. Let $H_1$ be the reflection of H in BC, so $H_1$ lies on the circumcircle. So area BCH = area $BCH_1$ <= area $BCA_1$. Adding to the two similar inequalities gives area ABC <= area hexagon - area ABC.
Alternative solution:
CAB = 180 - $CA_1B$ and $A_1B = A_1C$, so $A_1BC$ = 90 - 1/2 $CA_1B$ = 1/2 CAB. Hence the perpendicular from $A_1$ to BC has length 1/2 BC tan(CAB/2) and area $CA_1B$ = 1/4 $BC^2$ tan(CAB/2).
Put r = radius of in-circle of ABC, x = cot(CAB/2), y = cot(ABC/2), z = cot(BCA/2). Then BC = r(y + z) and area $CA_1B = r^2(y + z)^2/(4x)$. Also area BIC = 1/2 r BC. Similarly for the other triangles, so area ABC = area BIC + area CIA + area AIB = $r^2(x + y + z)$. We have to show that area ABC <= area $CA_1B$ + area $AB_1C$ + area $BC_1A$, or $(x + y + z) <= (y + z)^2/(4x) + (z + x)^2/(4y) + (x + y)^2/(4z)$.
Putting s = x + y + z, this is equivalent to: $4s <= (s - x)^2/x + (s - y)^2/y + (s - z)^2/z$, or $9s <= s^2(1/x + 1/y + 1/z)$, but this is just the statement that the arithmetic mean of x, y, z is not less than the harmonic mean.
Note in passing that the requirement for ABC to be acute is unnecessary.

**Problem 3**
**Solution** Three variants on a theme, all kindly supplied by others (I spent 2 hours failing to solve it). My favorite first.

Consider the pairs P, {A, B}, where P, A, B are points of S, and P lies on the perpendicular bisector of AB. There are at least n k(k - 1)/2 such pairs, because for each point P, there are at least k points equidistant from P and hence at least k(k - 1)/2 pairs of points equidistant from P.
If k >= 1/2 + √(2n), then k(k - 1) >= 2n - 1/4 > 2(n - 1), and so there are more than n(n - 1) pairs P, {A, B}. But there are only n(n - 1)/2 possible pairs {A, B}, so for some {$A_0$, $B_0$} we must be able to find at least 3 points P on the perpendicular bisector of $A_0B_0$. But these points are collinear, contradicting the assumption in the question.

*From an anonymous source*
Let the points be $P_1, P_2, \dots, P_n$. Let $C_i$ be a circle center $P_i$ containing at least k points of S. There are at least nk pairs $(C_i,P_j)$, where $P_j$ lies on $C_i$. Hence there must a point P

lying on at least k circles. Take k such circles $C_\alpha$. For each such circle $C_\alpha$, take a subset $S_\alpha$ comprising exactly k points of $S \cap C_\alpha$.
We now count the points in $\cup S_\alpha$. Apart from P, there are k-1 points in each $S_\alpha$. So we start with 1 + k(k-1). But this counts some points more than once. Each pair $(S_\alpha, S_\beta)$ (with $\alpha$ not equal to $\beta$) has at most one common point apart from P (because distinct circles have at most two common points). So we deduct 1 for each of the 1/2 k(k - 1) pairs $(S_\alpha, S_\beta)$, giving 1 + k(k - 1) - 1/2 k(k - 1) (*).
If Q (not equal to P) is in exactly r sets $S_\alpha$, then it is counted r times in the second term k(k - 1), and subtracted 1/2 r(r - 1) times in the third term. So it is counted 1/2 r(3 - r) times in all. That is correct for r = 0, 1 or 2 and too low for r > 2. So (*) is <= $|\cup S_\alpha|$.
Clearly $|\cup S_\alpha|$ <= n, so n >= 1 + k(k - 1)/2 = $(k - 1/2)^2/2 + 7/8 > (k - 1/2)^2/2$. Hence $\sqrt{(2n)} + 1/2 > k$.

*Alternative solution.*
Define $P_i$and $S_i$ as above. Let g(i) be the number of $S_a$ containing $P_i$, and let f(i, j) be $|S_i \cap S_j|$. Let h(x) = x(x - 1)/2. We count the number N of pairs i, {a, b}, where point i is in $S_a$ and $S_b$.
Point i is in g(i) sets $S_j$, from which we can choose $S_a$,$S_b$ in h(g(i)) ways. Hence N = $\Sigma$ h(g(i)). But f(a, b) points are in $S_a$ and $S_b$, so N = $\Sigma$ f(a, b). But, since distinct circles intersect in at most 2 points, f(a, b) <= 2, so $\Sigma$ f(a, b) <= h(n) 2. We conclude that 2 h(n) >= $\Sigma$ h(g(i)).
h is a convex function, so 1/n $\Sigma$h(g(i)) >= h(1/n $\Sigma$ g(i)) = n h(1/n nk) = n h(k). Hence n - 1 >= h(k), which gives the result, as above.

**Problem 4**
**Solution**
Let $C_A$ be the circle center A, radius AD, and $C_B$ the circle center B, radius BC. The circles touch on AB. Let $C_P$ the the circle center P, radius h. $C_P$ touches $C_A$ and $C_B$ and CD. Let t be the common tangent to $C_A$ and $C_B$ whose two points of contact are on the same side of AB as C and D. Then $C_P$ is confined inside the curvilinear triangle whose sides are segments of t, $C_A$ and $C_B$. Evidently h attains its maximum value, for given lengths AB, AD, BC, when $C_P$ touches t, in which case D must be the point at which t touches $C_A$, and C the point at which it touches $C_B$. Suppose E is the point at which t touches $C_P$.
Angles ADC and ACD are right angles, so $CD^2 = AB^2 - (AD - BC)^2 = 4$ AD BC. Similarly, $DE^2 = 4$ h AD, and $CE^2 = 4$ h BC. But CD = DE + CE, so $1/\sqrt{h} = 1/\sqrt{AD} + 1/\sqrt{BC}$. This gives the maximum value of h, so in general we have the inequality stated.

**Problem 5**
**Solution** Consider $(N!)^2+2, (N!)^2+3, \dots, (N!)^2+N$. $(N!)^2+r$ is divisible by r, but $((N!)^2+r)/r = N! (N!/r) + 1$, which is greater than one, but relatively prime to r since N! (N!/r) is divisible by r. For each r we may take a prime $p_r$ dividing r, so $(N!)^2+r$ is divisible by $p_r$, but is not a power of $p_r$. Hence it is not a prime or a prime power. Taking N = n+1 gives n consecutive numbers as required.

**Problem 6**
**Solution.**





Let $A_k$ be the set of permutations with k and k+n in neighboring positions, and let A be the set of permutations with property P, so that A is the union of the $A_k$.
Then $|A| = \text{Sum}_k |A_k| - \text{Sum}_{k<l} |A_k \cap A_l| + \text{Sum}_{k<l<m} |A_k \cap A_l \cap A_m| - \dots$ . But this is an alternating sequence of monotonically decreasing terms, hence $|A| >= \text{Sum}_k |A_k| - \text{Sum}_{k<l} |A_k \cap A_l|$.
But $|A_k|$ = 2 (2n - 1)! (two orders for k, k+n and then (2n - 1)! ways of arranging the 2n - 1 items, treating k, k+n as a single item). Similarly, $|A_k \cap A_l|$ = 4 (2n - 2)! So |A| >= (2n - 2)! [n.2(2n -1) - n(n - 1)/2 4] = $2n^2$ (2n - 2)! > (2n)!/2.

## 1990

**Problem 1**
**Solution** Angle ECG = angle DCA = angle DBA (since ACBD is cyclic) = angle DBM = angle MDB. Also angle CEG = 180 - angle EMD = angle BMD. So triangles CEG and BMD are similar.
Angle CEF = angle EMD = angle AMD. Also angle ECF = angle BCD = angle BAD (since BCAD is cyclic) = angle MAD. So triangles CEF and AMD are similar.
Hence EG/CE = MD/BM, EF/CE = MD/AM, and so dividing, EG/EF = AM/BM = t/(1-t).

**Problem 2**
**Solution** Answer: n for n ≡ 0 or 1 (mod 3), n - 1 for n ≡ 2 (mod 3).
Label the points 1 to 2n - 1. Two points have exactly n points between them if their difference (mod 2n - 1) is n - 2 or n + 1. We consider separately the three cases n = 3m, 3m + 1 and 3m + 2.
Let n = 3m. First, we exhibit a bad coloring with n - 1 black points. Take the black points to be 1, 4, 7, ... , 6m - 2 (2m points) and 2, 5, 8, ... , 3m - 4 (m - 1 points). It is easy to check that this is bad. The two points which could pair with r to give n points between are r + 3m - 2 and r + 3m + 1. Considering the first of these, 1, 4, 7, ... , 6m - 2 would pair with 3m - 1, 3m + 2, 3m + 5, ... , 6m - 1, 3, 6, ... , 3m - 6, none of which are black. Considering the second, they would pair with 3m + 2, 3m + 5, ... , 6m - 1, 3, ... , 3m - 3, none of which are black. Similarly, 2, 5, 8, ... , 3m - 4 would pair with 3m, 3m + 3, ... , 6m - 3, none of which are black. So the set is bad.
Now if we start with 1 and keep adding 3m - 2, reducing by 6m - 1 when necessary to keep the result in the range 1, ... , 6m - 1, we eventually get back to 1: 1, 3m - 1, 6m - 3, 3m - 4, 6m - 6, ... , 2, 3m, 6m - 2, 3m - 3, 6m - 5, ... , 3, 3m + 1, 6m - 1, ... , 4, 3m + 2, 1. The sequence includes all 6m - 1 numbers. Moreover a bad coloring cannot have any two consecutive numbers colored black. But this means that at most n - 1 out of the 2n - 1 numbers in the sequence can be black. This establishes the result for n = 3m.
Take n = 3m + 1. A bad coloring with n - 1 black points has the following black points: 1, 4, 7, ... , 3m - 2 (m points) and 2, 5, 8, ... , 6m - 1 (2m points). As before we add n - 2 repeatedly starting with 1 to get: 1, 3m, 6m - 1, 3m - 3, 6m - 4, ... , 3, 3m + 2, 6m + 1, 3m - 1, ... , 2, 3m + 1, 6m, 3m - 2, ... , 1. No two consecutive numbers can be black in a bad set, so a bad set can have at most n - 1 points.
Finally, take n = 3m + 2. A bad coloring with n - 2 points is 1, 2, ... , n - 2. This time when we add n - 2 = 3m repeatedly starting with 1, we get back to 1 after including only one-third of the numbers: 1, 3m + 1, 6m + 1, 3m - 2, ... , 4, 3m + 4, 1. The usual argument shows that at most m of these 2m + 1 numbers can be colored black in a bad set. Similarly, we may add 3m repeatedly starting with 2 to get another 2m + 1

numbers: 2, 3m + 2, 6m + 2, 3m - 1, ... , 3m + 5, 2. At most m of these can be black in a bad set. Similarly at most m of the 2m + 1 numbers: 3, 3m + 3, 6m + 3, 3m, ... , 3m + 6, 3 can be black. So in total at most 3m = n - 2 can be black in a bad set.

**Problem 3**
**Solution** Answer: n = 3.
Since $2^n + 1$ is odd, n must also be odd. Let p be its smallest prime divisor. Let x be the smallest positive integer such that $2^x = -1$ (mod p), and let y be the smallest positive integer such that $2^y = 1$ (mod p). y certainly exists and indeed y < p, since $2^{p-1} = 1$ (mod p). x exists since $2^n = -1$ (mod p). Write n = ys + r, with 0 <= r < y. Then $-1 = 2^n = (2^y)^s 2^r = 2^r$ (mod p), so x <= r < y (r cannot be 0, since - 1 is not 1 (mod p) ).
Now write n = hx + k, with 0 <= k < x. Then $-1 = 2^n = (-1)^h 2^k$ (mod p). Suppose k > 0. Then if h is odd we contradict the minimality of y, and if h is even we contradict the minimality of x. So k = 0 and x divides n. But x < p and p is the smallest prime dividing n, so x = 1. Hence 2 = -1 (mod p) and so p = 3.
Now suppose that $3^m$ is the largest power of 3 dividing n. We show that m must be 1. Expand $(3 - 1)^n + 1$ by the binomial theorem, to get (since n is odd): 1 - 1 + n.3 - 1/2 n(n - 1) $3^2$ + ... = 3n - (n - 1)/2 n $3^2$ + ... . Evidently 3n is divisible by $3^{m+1}$, but not $3^{m+2}$. We show that the remaining terms are all divisible by $3^{m+2}$. It follows that $3^{m+1}$ is the highest power 3 dividing $2^n + 1$. But $2^n + 1$ is divisible by $n^2$ and hence by $3^{2m}$, so m must be 1.
The general term is $(3^m a)$Cb $3^b$, for b >= 3. The binomial coefficients are integral, so the term is certainly divisible by $3^{m+2}$ for b >= m+2. We may write the binomial coefficient as $(3^m a/b)$ $(3^m - 1)/1$ $(3^m - 2)/2$ $(3^m - 3)/3$ ... $(3^m - (b-1))$ / (b - 1). For b not a multiple of 3, the first term has the form $3^m$ c/d, where 3 does not divide c or d, and the remaining terms have the form c/d, where 3 does not divide c or d. So if b is not a multiple of 3, then the binomial coefficient is divisible by $3^m$, since b > 3, this means that the whole term is divisible by at least $3^{m+3}$. Similarly, for b a multiple of 3, the whole term has the same maximum power of 3 dividing it as $3^m$ $3^b$/b. But b is at least 3, so $3^b$/b is divisible by at least 9, and hence the whole term is divisible by at least $3^{m+2}$.
We may check that n = 3 is a solution. If n > 3, let n = 3 t and let q be the smallest prime divisor of t. Let w be the smallest positive integer for which $2^w = -1$ (mod q), and v the smallest positive integer for which $2^v = 1$ (mod q). v certainly exists and < q since $2^{q-1}$ = 1 (mod q). $2^n = -1$ (mod q), so w exists and, as before, w < v. Also as before, we conclude that w divides n. But w < q, the smallest prime divisor of n, except 3. So w = 1 or 3. These do not work, because then 2 = -1 (mod q) and so q = 3, or $2^3 = -1$ (mod q) and again q =3, whereas we know that q > 3.

**Problem 4**
**Solution** We show first that f(1) = 1. Taking x = y = 1, we have f(f(1)) = f(1). Hence f(1) = f(f(1)) = f(1 f(f(1)) ) = f(1)/f(1) = 1.
Next we show that f(xy) = f(x)f(y). For any y we have 1 = f(1) = f(1/f(y) f(y)) = f(1/f(y))/y, so if z = 1/f(y) then f(z) = y. Hence f(xy) = f(xf(z)) = f(x)/z = f(x) f(y).
Finally, f(f(x)) = f(1 f(x)) = f(1)/x = 1/x.
We are not required to find all functions, just one. So divide the primes into two infinite sets S = {$p_1$, $p_2$, ... } and T= {$q_1$, $q_2$, ... }. Define $f(p_n) = q_n$, and $f(q_n) = 1/p_n$. We extend this definition to all rationals using f(xy) = f(x) f(y): $f(p_{i1}p_{i2}...q_{j1}q_{j2}.../(p_{k1}...q_{m1}...)) = p_{m1}...q_{i1}.../(p_{j1}...q_{k1}...)$. It is now trivial to verify that f(x f(y)) = f(x)/y.





**Problem 5**
**Solution** Answer: if $n_0$ = 2, 3, 4 or 5 then A loses; if $n_0$ >= 8, then A wins; if $n_0$ = 6 or 7 , then it is a draw.
A's strategy given a number n is as follows:
(1) if n ∈ [8, 11], pick 60
(2) if n ∈ [12, 16], pick 140
(3) if n ∈ [17, 22], pick 280
(4) if n ∈ [23, 44], pick 504
(5) if n ∈ [45, 1990], pick 1990
(6) if n = 1991 = 11.181 (181 is prime), pick 1991
(7) if n ∈ $[11^r 181 + 1, 11^{r+1} 181]$ for some r > 0, pick $11^{r+1}181$.
Clearly (5) wins immediately for A. After (4) B has 7.8.9 so must pick 56, 63, 72 or 168, which gives A an immediate win by (5). After (3) B must pick 35, 40, 56, 70 or 140, so A wins by (4) and (5). After (2) B must pick 20, 28, 35 or 70, so A wins by (3) - (5). After (1) B must pick 12, 15, 20 or 30, so A wins by (2) - (5).
If B is given $11^{r+1}181$, then B must pick 181, 11.181, ... , $11^r$.181 or $11^{r+1}$, all of which are <= $11^r$.181. So if A is given a number n in (6) or (7) then after a turn each A is given a number < n (and >= 11), so after a finite number of turns A wins.
If B gets a number less than 6, then he can pick 1 and win. Hence if A is given 2, he loses, because he must pick a number less than 5. Now if B gets a number of 11 or less, he wins by picking 1 or 2. Hence if A is given 3, he loses, because he must pick a number less than 10. Now if B gets a number of 19 or less, he can win by picking 1, 2 or 3. So if A is given 4 he loses. Now if B is given 29 or less, he can pick 1, 2, 3 or 4 and win. So if A is given 5 he loses.
We now have to consider what happens if A gets 6 or 7. He must pick 30 or more, or B wins. If he picks 31, 32, 33, 34, 35 or 36, then B wins by picking (for example) 1, 1, 3, 2, 5, 4 respectively. So his only hope given 6 is to pick 30. B also wins given any of 37, 38, 39, 40, 41, 43, 44, 45, 46, 47, 48, 49 (winning moves, for example, 37, 1; 38, 2; 39, 3; 40, 5; 41, 1; 43, 1; 44, 4; 45, 5; 46, 3; 47, 1; 48, 3]. So A's only hope given 7 is to pick 30 or 42.
If B is faced with 30=2.3.5, then he has a choice of 6, 10, 15. We have already established that 10 and 15 will lose, so he must pick 6. Thus 6 is a draw: A must pick 30 or lose, and then B must pick 6 or lose.
If B is faced with 42=2.3.7, then he has a choice of 6, 14 or 21. We have already established that 14 and 21 lose, so he must pick 6. Thus 7 is also a draw: A must pick 30 or 42, and then B must pick 6.

Comment
I am grateful to Gerhard Woeginger and Jean-Pierre Ehrmann for finding errors in my original solution.

**Problem 6**
**Solution** In the complex plane we can represent the sides as $p_n^2 w^n$, where $p_n$ is a permutation of (1, 2, ... , 1990) and w is a primitive 1990th root of unity.
The critical point is that 1990 is a product of more than 2 distinct primes: 1990 = 2.5.199. So we can write w = -1.a.b, where -1 is primitive 2nd root of unity, a is a primitive 5th root of unity, and b is a primitive 199th root of unity.
Now given one of the 1990th roots we may write it as $(-1)^i a^j b^k$, where 0 < i < 2, 0 < j < 5, 0 < k < 199 and hence associate it with the integer r(i,j,k) = 1 + 995i + 199j + k. This

is a bijection onto (1, 2, ... , 1990). We have to show that the sum of $r(i,j,k)^2 (-1)^i a^j b^k$ is zero.
We sum first over i. This gives $-995^2$ x sum of $a^j b^k$ which is zero, and - 1990 x sum $s(j,k) a^j b^k$, where s(j,k) = 1 + 199j + k. So it is sufficient to show that the sum of s(j,k) $a^j b^k$ is zero. We now sum over j. The 1 + k part of s(j,k) immediately gives zero. The 199j part gives a constant times $b^k$, which gives zero when summed over k.

## 1991

**Problem 1**
**Solution** Consider the areas of the three triangles ABI, BCI, CAI. Taking base BC we conclude that (area ABI + area CAI)/area ABC = AI/AA'. On the other hand, if r is the radius of the in-circle, then area ABI = AB.r/2 and similarly for the other two triangles. Hence AI/AA' = (CA + AB)/p, where p is the perimeter. Similarly BI/BB' = (AB + BC)/p and CI/CC' = (BC + CA)/p. But the arithmetic mean of (CA + AB)/p, (AB + BC)/p and (BC + CA)/p is 2/3. Hence their product is at most $(2/3)^3 = 8/27$.
Let AB + BC - CA = 2z, BC + CA - AB = 2x, CA + AB - BC = 2y. Then x, y, z are all positive and we have AB = y + z, BC = z + x, CA = x + y. Hence (AI/AA')(BI/BB')(CI/CC') = (1/2 + y/p)(1/2 + z/p)(1/2 + x/p) > 1/8 + (x+y+z)/(4p) = 1/8 + 1/8 = 1/4.

**Problem 2**
**Solution** If n is odd, then 1 and 2 are prime to n, so all integers < n are prime to n, and hence is prime.
If n = 4k, then 2k-1 and 2k+1 are prime to n, so all odd integers < n are prime to n, and hence n must be a power of 2.
If n = 4k+2, then 2k+1 divides n, but 2k+3 and 2k+5 are prime to n. But if n > 6, then 2k+5 < n, so this cannot be a solution.

**Problem 3**
**Solution** Answer: 217.
Let A be the subset of all multiples of 2, 3, 5 or 7. Then A has 216 members and every 5-subset has 2 members with a common factor. [To show that |A| = 216, let $a_n$ be the number of multiples of n in S. Then $a_2 = 140$, $a_3 = 93$, $a_5 = 56$, $a_6 = 46$, $a_{10} = 28$, $a_{15} = 18$, $a_{30} = 9$. Hence the number of multiples of 2, 3 or 5 = $a_2 + a_3 + a_5 - a_6 - a_{10} - a_{15} + a_{30} = 206$. There are ten additional multiples of 7: 7, 49, 77, 91, 119, 133, 161, 203, 217, 259.]
Let P be the set consisting of 1 and all the primes < 280. Define:
A1 = {2.41, 3.37, 5.31, 7.29, 11.23, 13.19}
A2 = {2.37, 3.31, 5.29, 7.23, 11.19, 13.17}
A3 = {2.31, 3.29, 5.23, 7.19, 11.17, 13.13}
B1 = {2.29, 3.23, 5.19, 7.17, 11.13}
B2 = {2.23, 3.19, 5.17, 7.13, 11.11}
Note that these 6 sets are disjoint subsets of S and the members of any one set are relatively prime in pairs. But P has 60 members, the three As have 6 each, and the two Bs have 5 each, a total of 88. So any subset T of S with 217 elements must have at least 25 elements in common with their union. But 6.4 = 24 < 25, so T must have at least 5 elements in common with one of them. Those 5 elements are the required subset of elements relatively prime in pairs.





**Problem 4**
**Solution** The basic idea is that consecutive numbers are relatively prime.
We construct a labeling as follows. Pick any vertex A and take a path from A along unlabeled edges. Label the edges consecutively 1, 2, 3, ... as the path is constructed. Continue the path until it reaches a vertex with no unlabeled edges. Let B be the endpoint of the path. A is now guaranteed to have the gcd (= greatest common divisor) of its edges 1, because one of its edges is labeled 1. All the vertices between A and B are guaranteed to have gcd 1 because they have at least one pair of edges with consecutive numbers. Finally, either B has only one edge, in which case its gcd does not matter, or it is also one of the vertices between A and B, in which case its gcd is 1.
Now take any vertex C with an unlabeled edge and repeat the process. The same argument shows that all the new vertices on the new path have gcd 1. The endpoint is fine, because either it has only one edge (in which case its gcd does not matter) or it has already got gcd 1.
Repeat until all the edges are labeled.

**Problem 5**
**Solution** *By Marcin Mazur, University of Illinois at Urbana-Champaign*
Let P, Q, R be the feet of the perpendiculars from X to BC, CA, AB respectively. Use A, B, C to denote the interior angles of the triangle (BAC, CBA, ACB). We have PX = BX sin XBC = CX sin(C - XCA), QX = CX sin XCA = AX sin(A - XAB), RX = AX sin XAB = BX sin(B - XBC). Multiplying: sin(A - XAB) sin(B - XBC) sin(C - XCA) = sin A sin B sin C.
Now observe that sin(A - x)/sin x = sin A cot x - cos A is a strictly decreasing function of x (over the range 0 to $\pi$), so if XAB, XBC and XCA are all greater than 30, then sin(A - 30) sin(B - 30) sin(C - 30) > $\sin^3 30 = 1/8$.
But sin(A - 30) sin(B - 30) = (cos(A - B) - cos(A + B - 60))/2 <= (1 - cos(A + B - 60))/2 = (1 - sin(C - 30))/2, since (A - 30) + (B - 30) + (C - 30) = 90. Hence sin(A - 30) sin(B - 30) sin(C - 30) <= 1/2 (1 - sin(C - 30)) sin(C - 30) = 1/2 (1/4 - $(\sin(C - 30) - 1/2)^2$) <= 1/8. So XAB, XBC, XCA cannot all be greater than 30.

*By Jean-Pierre Ehrmann*
P, Q, R as above. Area ABX + area BCX + area CAX = area ABC, so AB.XR + BC.XP + CA.XQ = 2 area ABC <= BC.AP <= BC(AX + XP). Hence AB.XR/AX + CA.XQ/AX <= BC.
Squaring and using $(\lambda + \mu)^2 >= 4 \lambda\mu$, we have: $BC^2 >= 4$ AB.CA. XR.XQ/$AX^2$.
Similarly: $CA^2 >= 4$ BC.AB.XP.XR/$BX^2$, and $AB^2 >= 4$ AB.BC.XQ.XP/$CX^2$.
Multiplying these three inequalities together gives: 1 >= 64 $(XR/AX)^2(XP/BX)^2(XQ/CX)^2$, and hence: (XR/AX) (XP/BX) (XQ/CX) <= 1/8, or sin XAB sin XBC sin XCA <= 1/8. So not all XAB, XBC, XCA are greater than 30.

*Gerard Gjonej noted that the result follows almost immediately from the Erdos-Mordell inequality: XA + XB + XC >= 2(XP + XQ + XR). [For if all the angles are greater than 30, then XR/XA, XP/XB, XQ/XC are all greater than sin 30 = 1/2.]. This result was notoriously hard to prove - Erdos hawked it around a large number of mathematicians before Mordell found a proof - but the proof now appears fairly innocuous, at least if you do not have to rediscover it:*

Let $R_1$, $Q_1$ be the feet of the perpendiculars from P to AB, CA respectively. Similarly, let $P_2$, $R_2$ be the feet of the perpendiculars from Q to BC, AB, and $Q_3$, $P_3$ the feet of the perpendiculars from R to CA, BC. Then $P_2P_3$ is the projection of QR onto BC, so $P_2P_3/QR \le 1$. Similarly, $Q_3Q_1/RP \le 1$, and $R_1R_2/PQ \le 1$. Hence $XA + XB + XC \ge XA.P_2P_3/QR + XB.Q_3Q_1/RP + XC.R_1R_2/PQ$ (*)
Now BPXR is cyclic, because BPX and XRB are both right angles. Hence angle BXR = angle BPR = angle $RPP_3$, so triangles XBR and $PRP_3$ are similar. Hence $PP_3 = PR.XR/XB$.
Similarly, $QQ_1 = QP.XP/XC$, $RR_2 = RQ.XQ/XA$, and $PP_2 = PQ.XQ/XC$, $QQ_3 = QR.XR/XA$, $RR_1 = RP.XP/XB$. Substituting into (*), we obtain:
XA + XB + XC >= XA( PQ/QR XQ/XC + PR/QR XR/XB ) + XB( QR/RP XR/XA + QP/RP XP/XC ) + XC( RP/PQ XP/XB + RQ/PQ XQ/XA ).
On the right hand side, the terms involving XP are: XP( QP/RP XB/XC + RP/PQ XC/XB ), which has the form XP (x + 1/x) and hence is at least 2 XP. Similarly for the terms involving XQ and XR.

**Problem 6**
**Solution** Let $t = 1/2^a$. Define $c = 1 - t/(1 - t)$. Since $a > 1$, $c > 0$. Now given any integer $n > 0$, take the binary expansion $n = \Sigma_i b_i 2^i$, and define $x_n = 1/c \Sigma_{bi>0} t^i$. For example, taking $n = 21 = 2^4 + 2^2 + 2^0$, we have $x_{21} = (t^4 + t^2 + t^0)/c$. We show that for any unequal n, m, $|x_n - x_m| |n - m|^a \ge 1$. This solves the problem, since the $x_n$ are all positive and bounded by $(\Sigma t^n )/c = 1/(1 - 2t)$.
Take k to be the highest power of 2 dividing both n and m. Then $|n - m| \ge 2^k$. Also, in the binary expansions for n and m, the coefficients of $2^0, 2^1, \dots , 2^{k-1}$ agree, but the coefficients for $2^k$ are different. Hence $c |x_n - x_m| = t^k + \Sigma_{i>k} y_i$, where $y_i = 0$, $t^i$ or $- t^i$. Certainly $\Sigma_{i>k} y_i > - \Sigma_{i>k} t^i = t^{k+1}/(1 - t)$, so $c |x_n - x_m| > t^k(1 - t/(1 - t)) = c t^k$. Hence $|x_n - x_m| |n - m|^a > t^k 2^{ak} = 1$.

## 1992

**Problem 1**
**Solution** Answer: a = 2, b = 4, c = 8; or a = 3, b = 5, c = 15.
Let $k = 2^{1/3}$. If $a \ge 5$, then $k(a - 1) > a$. [Check: $(k(a - 1))^3 - a^3 = a^3 - 6a^2 + 6a - 2$. For $a \ge 6$, $a^3 \ge 6a^2$ and $6a > 2$, so we only need to check a = 5: 125 - 150 + 30 - 2 = 3.] We know that c > b > a, so if a >= 5, then 2(a - 1)(b - 1)(c - 1) > abc > abc - 1. So we must have a = 2, 3 or 4.
Suppose abc - 1 = n(a - 1)(b - 1)(c - 1). We consider separately the cases n = 1, n = 2 and n >= 3. If n = 1, then a + b + c = ab + bc + ca. But that is impossible, because a, b, c are all greater than 1 and so a < ab, b < bc and c < ca.
Suppose n = 2. Then abc - 1 is even, so all a, b, c are odd. In particular, a = 3. So we have 4(b - 1)(c - 1) = 3bc - 1, and hence bc + 5 = 4b + 4c. If b >= 9, then bc >= 9c > 4c + 4b. So we must have b = 5 or 7. If b = 5, then we find c = 15, which gives a solution. If b = 7, then we find c = 23/3 which is not a solution.
The remaining case is n >= 3. If a = 2, we have n(bc - b - c + 1) = 2bc - 1, or (n - 2)bc + (n + 1) = nb + nc. But b >= 3, so (n - 2)bc >= (3n - 6)c >= 2nc for n >= 6, so we must have n = 3, 4 or 5. If n = 3, then bc + 4 = 3b + 3c. If b >= 6, then bc >= 6c > 3b + 3c, so b = 3, 4 or 5. Checking we find only b = 4 gives a solution: a = 2, b = 4, c = 8. If n = 4, then (n - 2)bc, nb and nc are all even, but (n + 1) is odd, so there is no solution. If n = 5,





then 3bc + 6 = 5b + 5c. b = 3 gives c = 9/4, which is not a solution. b >= 4 gives 3bc > 10c > 5b + 5c, so there are no solutions.
If a = 3, we have 2n(bc - b - c + 1) = 3bc - 1, or (2n - 3)bc + (2n + 1) = 2nb + 2nc. But b >= 4, so (2n - 3)bc >= (8n - 12)c >= 4nc > 2nc + 2nb. So there are no solutions.
Similarly, if a = 4, we have (3n - 4)bc + (3n + 1) = 3nb + 3nc. But b >= 4, so (3n - 4)bc >= (12n - 16)c > 6nc > 3nb + 3nc, so there are no solutions.

**Problem 2**
**Solution** The first step is to establish that f(0) = 0. Putting x = y = 0, and f(0) = t, we get $f(t) = t^2$. Also, $f(x^2+t) = f(x)^2$, and $f(f(x)) = x + t^2$. We now evaluate $f(t^2+f(1)^2)$ two ways. First, it is $f(f(1)^2 + f(t)) = t + f(f(1))^2 = t + (1 + t^2)^2 = 1 + t + 2t^2 + t^4$. Second, it is $f(t^2 + f(1 + t)) = 1 + t + f(t)^2 = 1 + t + t^4$. So t = 0, as required.
It follows immediately that f(f(x)) = x, and $f(x^2) = f(x)^2$. Given any y, let z = f(y). Then y = f(z), so $f(x^2 + y) = z + f(x)^2 = f(y) + f(x)^2$. Now given any positive x, take z so that $x = z^2$. Then $f(x + y) = f(z^2 + y) = f(y) + f(z)^2 = f(y) + f(z^2) = f(x) + f(y)$. Putting y = -x, we get 0 = f(0) = f(x + -x) = f(x) + f(-x). Hence f(-x) = - f(x). It follows that f(x + y) = f(x) + f(y) and f(x - y) = f(x) - f(y) hold for all x, y.
Take any x. Let f(x) = y. If y > x, then let z = y - x. f(z) = f(y - x) = f(y) - f(x) = x - y = -z. If y < x, then let z = x - y and f(z) = f(x - y) = f(x) - f(y) = y - x. In either case we get some z > 0 with f(z) = -z < 0. But now take w so that $w^2 = z$, then $f(z) = f(w^2) = f(w)^2 \ge 0$. Contradiction. So we must have f(x) = x.

**Problem 3**
**Solution** We show that for n = 32 we can find a coloring without a monochrome triangle. Take two squares $R_1R_2R_3R_4$ and $B_1B_2B_3B_4$. Leave the diagonals of each square uncolored, color the remaining edges of R red and the remaining edges of B blue. Color blue all the edges from the ninth point X to the red square, and red all the edges from X to the blue square. Color $R_iB_j$ red if i and j have the same parity and blue otherwise. Clearly X is not the vertex of a monochrome square, because if XY and XZ are the same color then, YZ is either uncolored or the opposite color. There is no triangle within the red square or the blue square, and hence no monochrome triangle. It remains to consider triangles of the form $R_iR_jB_k$ and $B_iB_jR_k$. But if i and j have the same parity, then $R_iR_j$ is uncolored (and similarly $B_iB_j$), whereas if they have opposite parity, then $R_iB_k$ and $R_jB_k$ have opposite colors (and similarly $B_iR_k$ and $B_jR_k$).
It remains to show that for n = 33 we can always find a monochrome triangle. There are three uncolored edges. Take a point on each of the uncolored edges. The edges between the remaining 6 points must all be colored. Take one of these, X. At least 3 of the 5 edges to X, say XA, XB, XC must be the same color (say red). If AB is also red, then XAB is monochrome. Similarly, for BC and CA. But if AB, BC and CA are all blue, then ABC is monochrome.

**Problem 4**
**Solution** Answer: Let X be the point where C meets L, let O be the center of C, let XO cut C gain at Z, and take Y on QR so that M be the midpoint of XY. Let L' be the line YZ. The locus is the open ray from Z along L' on the opposite side to Y.

Let C' be the circle on the other side of QR to C which also touches the segment QR and the lines PQ and QR. Let C' touch QR at Y'. If we take an expansion (technically, homothecy) center P, factor PY'/PZ, then C goes to C', the tangent to C at Z goes to the line QR, and hence Z goes to Y'. But it is easy to show that QX = RY'.

We focus on the QORO'. Evidently X,Y' are the feet of the perpendiculars from O, O' respectively to QR. Also, OQO' = ORO' = 90. So QY'O' and OXQ are similar, and hence QY'/Y'O' = OX/XQ. Also RXO and O'Y'R are similar, so RX/XO = O'Y'/Y'R. Hence QY'.XQ = OX.O'Y' = RX.Y'R. Hence QX/RX = QX/(QR - QX) = RY'/(QR - RY') = RY'/QY'. Hence QX = RY'.
But QX = RY by construction (M is the midpoint of XY and QR), so Y = Y'. Hence P lies on the open ray as claimed. Conversely, if we take P on this ray, then by the same argument QX = RY. But M is the midpoint of XY, so M must also be the midpoint of QR, so the locus is the entire (open) ray.
Take Cartesian coordinates with origin X, so that M is (a, 0) and O is (0, R). Let R be the point (b, 0) (we take a, b >= 0). Then Q is the point (2a - b, 0), and Y is (2a, 0). Let angle XRO be θ. Then tan θ = R/b and angle PRX = 2θ, so tan PRX = 2 tan θ/( 1 - $\tan^2\theta$) = 2Rb/($b^2$ - $R^2$). Similarly, tan PQX = 2R(b - 2a)/( $(b - 2a)^2$ - $R^2$).
If P has coordinates (A, B), then B/(b - A) = tan PRX, and B/(b - 2a + x) = tan PQX. So we have two simultaneous equations for A and B. Solving, and simplifying slightly, we find A = -2a$R^2$/($b^2$ - 2ab - $R^2$), B = 2b(b - 2a)R/($b^2$ - 2ab - $R^2$). (*)
We may now check that B/(2a - A) = R/a, so P lies on YZ as claimed. So we have shown that the locus is a subset of the line YZ. But since $b^2$ - 2ab - $R^2$ maps the open interval (a + $\sqrt{a^2 + R^2}$, ∞) onto the open interval (0, ∞), (*) shows that we can obtain any value A in the open interval (-∞,0) by a suitable choice of b, and hence any point P on the ray (except its endpoint Z) by a suitable choice of R.

**Problem 5**
**Solution** Induction on the number of different z-coordinates in S.
For 1, it is sufficient to note that S = $S_z$ and |S| <= $|S_x|\,|S_y|$ (at most $|S_x|$ points of S project onto each of the points of $S_y$).
In the general case, take a horizontal (constant z) plane dividing S into two non-empty parts T and U. Clearly, |S| = |T| + |U|, $|S_x| = |T_x| + |U_x|$, and $|S_y| = |T_y| + |U_y|$.
By induction, |S| = |T| + |U| <= $(|T_x|\,|T_y|\,|T_z|)^{1/2} + (|U_x|\,|U_y|\,|U_z|)^{1/2}$. But $|T_z|, |U_z| <= |S_z|$, and for any positive a, b, c, d we have $(a\,b)^{1/2} + (c\,d)^{1/2} <= ( (a + c)\,(b + d) )^{1/2}$ (square!). Hence $|S| <= |S_z|^{1/2}( ( |T_x| + |U_x| )\,( |T_y| + |U_y| ) )^{1/2} = ( |S_x|\,|S_y|\,|S_z| )^{1/2}$.

**Problem 6**
**Solution** (a) Let N = $n^2$. Suppose we could express N as a sum of N - 13 squares. Let the number of 4s be a, the number of 9s be b and so on. Then we have 13 = 3a + 8b + 15c + ... . Hence c, d, ... must all be zero. But neither 13 nor 8 is a multiple of 3, so there are no solutions. Hence S(n) <= N - 14.
A little experimentation shows that the problem is getting started. Most squares cannot be expressed as a sum of two squares. For N = $13^2$ = 169, we find: 169 = 9 + 4 + 4 + 152 1s, a sum of 155 = N - 14 squares. By grouping four 1s into a 4 repeatedly, we obtain all multiples of 3 plus 2 down to 41 (169 = 9 + 40 4s). Then grouping four 4s into a 16 gives us 38, 35, ... , 11 (169 = 10 16s + 9). Grouping four 16s into a 64 gives us 8 and 5. We obtain the last number congruent to 2 mod 3 by the decomposition: 169 = $12^2$ + $5^2$.
For the numbers congruent to 1 mod 3, we start with N - 15 = 154 squares: 169 = 5 4s + 149 1s. Grouping as before gives us all 3m + 1 down to 7: 169 = 64 + 64 + 16 + 16 + 4 + 4 + 1. We may use 169 = $10^2$ + $8^2$ + $2^2$ + $1^2$ for 4.
For multiples of 3, we start with N - 16 = 153 squares: 169 = 9 + 9 + 151 1s. Grouping as before gives us all multiples of 3 down to 9: 169 = 64 + 64 + 16 + 9 + 9 + 4 + 1 + 1 +





1. Finally, we may take 169 = $12^2$ + $4^2$ + $3^2$ for 3 and split the $4^2$ to get 169 = $12^2$ + $3^2$ + $2^2$ + $2^2$ + $2^2$ + $2^2$ for 6. That completes the demonstration that we can write $13^2$ as a sum of k positive squares for all k <= S(13) = $13^2$ - 14.
We now show how to use the expressions for $13^2$ to derive further N. For any N, the grouping technique gives us the high k. Simply grouping 1s into 4s takes us down: from 9 + 4 + 4 + (N-17) 1s to (N-14)/4 + 6 < N/2 or below; from 4 + 4 + 4 + 4 + 4 + (N-20) 1s to (N-23)/4 + 8 < N/2 or below; from 9 + 9 + (N-18) 1s to (N-21)/4 + 5 < N/2 or below. So we can certainly get all k in the range (N/2 to N-14) by this approach. Now suppose that we already have a complete set of expressions for $N_1$ and for $N_2$ (where we may have $N_1 = N_2$). Consider $N_3 = N_1N_2$. Writing $N_3 = N_1$( an expression for $N_2$ as a sum of k squares) gives $N_3$ as a sum of 1 thru $k_2$ squares, where $k_2 = N_2 - 14$ squares (since $N_1$ is a square). Now express $N_1$ as a sum of two squares: $n_1^2 + n_2^2$. We have $N_3$ = $n_1^2$(a sum of $k_2$ squares) + $n_2^2$(a sum of k squares). This gives $N_3$ as a sum of $k_2$ + 1 thru $2k_2$ squares. Continuing in this way gives $N_3$ as a sum of 1 thru $k_1k_2$ squares. But $k_i = N_i$ - 14 > 2/3 $N_i$, so $k_1k_2 > N_3/2$. So when combined with the top down grouping we get a complete set of expressions for $N_3$.
This shows that there are infinitely many squares N with a complete set of expressions, for example we may take N = the squares of 13, $13^2$, $13^3$, ... .

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## 1993

**Problem 1**
**Solution** Suppose f(x) = ($x^r + a_{r-1}x^{r-1} + ... + a_1x$ +/- 3)($x^s + b_{s-1}x^{s-1} + ... + b_1x$ +/- 1). We show that all the a's are divisible by 3 and use that to establish a contradiction.
First, r and s must be greater than 1. For if r = 1, then +/- 3 is a root, so if n is even, we would have 0 = $3^n$ +/- 5 $3^{n-1}$ + 3 = $3^{n-1}$( 3 +/- 5) + 3, which is false since 3 +/- 5 = 8 or -2. Similarly if n is odd we would have 0 = $3^{n-1}$(+/-3 + 5) + 3, which is false since +/-3 + 5 = 8 or 2. If s = 1, then +/- 1 is a root and we obtain a contradiction in the same way.
So r <= n - 2, and hence the coefficients of x, $x^2$, ... , $x^r$ are all zero. Since the coefficient of x is zero, we have: $a_1$ +/- $3b_1$ = 0, so $a_1$ is divisible by 3. We can now proceed by induction. Assume $a_1, ... , a_t$ are all divisible by 3. Then consider the coefficient of $x^{t+1}$. If s-1 >=t+1, then $a_{t+1}$ = linear combination of $a_1, ... , a_t$ +/- $3b_{t+1}$. If s-1 < t+1, then $a_{t+1}$ = linear combination of some or all of $a_1, ... , a_t$. Either way, $a_{t+1}$ is divisible by 3. So considering the coefficients of x, $x^2$, ... , $x^{r-1}$ gives us that all the a's are multiples of 3. Now consider the coefficient of $x^r$, which is also zero. It is a sum of terms which are multiples of 3 plus +/-1, so it is not zero. Contradiction. Hence the factorization is not possible.

**Problem 2**
**Solution** Take B' so that CB = CB', BCB' = 90 and B' is on the opposite side of BC to A. It is easy to check that ADB, ACB' are similar and DAC, BAB' are similar. Hence AB/BD = AB'/B'C and CD/AC = BB'/AB'. It follows that the ratio given is BB'/B'C which is √2.
Take XD the tangent to the circumcircle of ADC at D, so that XD is in the angle ADB. Similarly, take YD the tangent to the circumcircle BDC at D. Then ADX = ACD, BDY = BCD, so ADX + BDY = ACB and hence XDY = ADB - (ADX + BDY) = ADB - ACB = 90. In other words the tangents to the circumcircles at D are perpendicular. Hence, by symmetry (reflecting in the line of centers) the tangents at C are perpendicular.

*Theo Koupelis, University of Wisconsin, Marathon* provided a similar solution (about 10 minutes later!) taking the point B' so that BDB' = 90, BD = B'D and B'DA = ACB. DAC, B'AB are similar; and ABC, AB'D are similar.

**Problem 3**

**Solution** We show first that the game can end with only one piece if n is not a multiple of 3. Note first that the result is true for n = 2 or 4.

n=2
```
X X   . . X   . . X   . . .
X X   X X     . . X   . . .
                          X
```
n = 4
```
              X         X         X         X         X
X X X X   X X . X   X X . X   . . X X   . . X X   . X . .
X X X X   X X . X   . . X X   . . X X   X . X X   X . X X
X X X X   X X X X   X X X X   X X X X   . X X X   . X X X
X X X X   X X X X   X X X X   X X X X   . X X X   . X X X

    X         X         X
. X . .   . X . .   . X X .   . X . .   . X . .   . . . .
X X . .   . . X .   . . . .   . . X .   . X X .   . . X .
. X X X   . X X X   . X . X   . X . X   . . . X   . X . X
. X X X   . X X X   . X X X   . X X X   . . X X   . . X X

. . . .   . . . .   . . . .   . . . .
. . X X   . X . .   . . . .   . . . .
. X . .   . X . .   . . . .   . . . .
. . X .   . . X .   . X X .   . . . X
```
The key technique is the following three moves which can be used to wipe out three adjacent pieces on the border provided there are pieces behind them:
```
 X X X     X X .     X X .     X X X
 X X X     X X .     . . X     . . .
               X         X
```
We can use this technique to reduce (r + 3) x s rectangle to an r x s rectangle. There is a slight wrinkle for the last two rows of three:
```
 X X X X     X X . X     . . X X     . . X X     . . . X     . . . X
 X X X X     X X . X     . . X X     . . X X     . . . X     . . . X
 . . . X     . . X X     . . X X     . X . .     . X X .     . . . X
```
Thus we can reduce a square side 3n+2 to a 2 x (3n+2) rectangle. We now show how to wipe out the rectangle. First, we change the 2 x 2 rectangle at one end into a single piece alongside the (now) 2 x 3n rectangle:
```
  X X     . .     . .
  X X     . .     . .
         X X         X
```
Then we use the following technique to shorten the rectangle by 3:
```
 X X X     X . X     X . X     . . .     . . .
 X X X     X . X     X . X     . . .     . . .
 X         X X     X .       X X . X         . X
```
That completes the case of the square side 3n+2. For the square side 3n+1 we can use the technique for removing 3 x r rectangles to reduce it to a 4 x 4 square and then use the technique above for the 4 x 4 rectangle.

Finally, we use a parity argument to show that if n is a multiple of 3, then the square side n cannot be reduced to a single piece. Color the board with 3 colors, red, white and blue:
```
 R W B R W B R W B ...
```





```
 W B R W B R W B R ...
 B R W B R W B R W ...
 R W B R W B R W B ...
 ...
```
Let suppose that the single piece is on a red square. Let A be the number of moves onto a red square, B the number of moves onto a white square and C the number of moves onto a blue square. A move onto a red square increases the number of pieces on red squares by 1, reduces the number of pieces on white squares by 1, and reduces the number of pieces on blue squares by 1. Let n = 3m. Then there are initially m pieces on red squares, m on white and m on blue. Thus we have:

- A + B + C = m-1; A - B + C = m; A + B - C = m.

Solving, we get A = m, B = m - 1/2, C = m - 1/2. But the number of moves of each type must be integral, so it is not possible to reduce the number of pieces to one if n is a multiple of 3.

**Problem 4**

**Solution** The length of an altitude is twice the area divided by the length of the corresponding side. Suppose that BC is the longest side of the triangle ABC. Then m(ABC) = area ABC/BC. [If A = B = C, so that BC = 0, then the result is trivially true.] Consider first the case of X inside ABC. Then area ABC = area ABX + area AXC + area XBC, so m(ABC)/2 = area ABX/BC + area AXC/BC + area XBC/BC. We now claim that the longest side of ABX is at most BC, and similarly for AXC and XBC. It then follows at once that area ABX/BC <= area ABX/longest side of ABX = m(ABX)/2 and the result follows (for points X inside ABC).

The claim follows from the following lemma. If Y lies between D and E, then FY is less than the greater than FD and FE. Proof: let H be the foot of the perpendicular from F to DE. One of D and E must lie on the opposite side of Y to H. Suppose it is D. Then FD = FH/cos HFD > FH/cos HFY = FY. Returning to ABCX, let CX meet AB at Y. Consider the three sides of ABX. By definition AB <= BC. By the lemma AX is smaller than the larger of AC and AY, both of which do not exceed BC. Hence AX <= BC. Similarly BX <= BC.

It remains to consider X outside ABC. Let AX meet AC at O. We show that the sum of the smallest altitudes of ABY and BCY is at least the sum of the smallest altitudes of ABO and ACO. The result then follows, since we already have the result for X = O. The altitude from A in ABX is the same as the altitude from A in ABO. The altitude from X in ABX is clearly longer than the altitude from O in ABO (let the altitudes meet the line AB at Q and R respectively, then triangles BOR and BXQ are similar, so XQ = OR.BX/BO > OR). Finally, let k be the line through A parallel to BX, then the altitude from B in ABX either crosses k before it meets AX, or crosses AC before it crosses AX. If the former, then it is longer than the perpendicular from B to k, which equals the altitude from A to BO. If the latter, then it is longer than the altitude from B to AO. Thus each of the altitudes in ABX is longer than an altitude in ABO, so m(ABX) > m(ABO).

**Problem 5**

**Solution** Answer: yes. f(n) = [g*n + 1/2], where g = golden ratio = (1 + √5)/2 = 1.618 ...
Let g(n) = [g*n + 1/2]. Obviously g(1) = 2. Also g(n+1) = g(n) + 1 or g(n) + 2, so certainly g(n+1) > g(n).

Consider d(n) = g* [g*n + 1/2] + 1/2 - ( [g*n + 1/2] + n). We show that it is between 0 and 1. It follows immediately that g(g(n)) = g(n) + n, as required.

Certainly, [g*n + 1/2] > g*n - 1/2, so d(n) > 1 - g/2 > 0 (the n term has coefficient $g^2$ - g - 1 which is zero). Similarly, [g*n + 1/2] <= g*n + 1/2, so d(n) <= g/2 < 1, which completes the proof.
*I originally put up the much clumsier result following:*
Answer: yes. Take n = $b_rb_{r-1}$ ... $b_0$ in the Fibonacci base. Then f(n) = $b_rb_{r-1}$ ... $b_0$0. This satisfies the required conditions.
Let $u_0$ = 1, $u_1$ = 2, ... , $u_n$=$u_{n-1}$+$u_{n-2}$, ... be the Fibonacci numbers. We say n = $b_rb_{r-1}$ ... $b_0$ in the Fibonacci base if $b_r$ = 1, every other $b_i$ = 0 or 1, no two adjacent $b_i$ are non-zero, and n = $b_ru_r$ + ... + $b_0u_0$. For example, 28 = 1001010 because 28 = 21 + 5 + 2.
We have to show that every n has a unique expression of this type. We show first by induction that it has at least one expression of this type. Clearly that is true for n = 1. Take $u_r$ to be the largest Fibonacci number <= n. Then by induction we have an expression for n - $u_r$. The leading term cannot be $u_i$ for i > r - 2, for then we would have n >= $u_r$ + $u_{r-1}$ = $u_{r+1}$. So adding $u_r$ to the expression for n - $u_r$ gives us an expression of the required type for n, which completes the induction.
We show that $u_r$ + $u_{r-2}$ + $u_{r-4}$ + ... = $u_{r+1}$ - 1. Again we use induction. It is true for r = 1 and 2. Suppose it is true for r - 1, then $u_{r+1}$ + $u_{r-1}$ + ... = $u_{r+2}$ - $u_r$ + $u_{r-1}$ + $u_{r-3}$ + ... = $u_{r+2}$ - $u_r$ + $u_r$ - 1 = $u_{r+2}$ - 1. So it is true for r + 1. Hence it is true for all r. Now we can prove that the expression for n is unique. It is for n = 1. So assume it is for all numbers < n, but that the expression for n is not unique, so that we have n = $u_r$ + more terms = $u_s$ + more terms. If r = s, then the expression for n - $u_r$ is not unique. Contradiction. So suppose r > s. But now the second expression is at most $u_{s+1}$ - 1 which is less than $u_r$. So the expression for n must be unique and the induction is complete.
It remains to show that f satisfies the required conditions. Evidently if n = 1 = $u_0$, then f(n) = $u_1$ = 2, as required. If n = $u_{a1}$ + ... + $u_{ar}$, then f(n) = $u_{a1+1}$ + ... + $u_{ar+1}$ and f(f(n)) = $u_{a1+2}$ + ... + $u_{ar+2}$. So f(n) + n = ($u_{a1}$ + $u_{a1+1}$) + ... + ($u_{ar}$ + $u_{ar+1}$) = f(f(n)).

**Problem 6**
**Solution** (a) The process cannot terminate, because before the last move a single lamp would have been on. But the last move could not have turned it off, because the adjacent lamp was off. There are only finitely many states (each lamp is on or off and the next move can be at one of finitely many lamps), hence the process must repeat. The outcome of each step is uniquely determined by the state, so either the process moves around a single large loop, or there is an initial sequence of steps as far as state k and then the process goes around a loop back to k. However, the latter is not possible because then state k would have had two different precursors. But a state has only one possible precursor which can be found by toggling the lamp at the current position if the previous lamp is on and then moving the position back one. Hence the process must move around a single large loop, and hence it must return to the initial state.
(b) Represent a lamp by X when on, by - when not. For 4 lamps the starting situation and the situation after 4, 8, 12, 16 steps is as follows:
```
X X X X
- X - X
X - - X
- - - X
X X X -
```
On its first move lamp n-2 is switched off and then remains off until each lamp has had n-1 moves. Hence for each of its first n-1 moves lamp n-1 is not toggled and it retains its initial state. After each lamp has had n-1 moves, all of lamps 1 to n-2 are off. Finally over the next n-1 moves, lamps 1 to n-2 are turned on, so that all the lamps are on. We





show by induction on k that these statements are all true for n = $2^k$. By inspection, they are true for k = 2. So suppose they are true for k and consider 2n = $2^{k+1}$ lamps. For the first n-1 moves of each lamp the n left-hand and the n right-hand lamps are effectively insulated. Lamps n-1 and 2n-1 remain on. Lamp 2n-1 being on means that lamps 0 to n-2 are in just the same situation that they would be with a set of only n lamps. Similarly, lamp n-1 being on means that lamps n to 2n-2 are in the same situation that they would be with a set of only n lamps. Hence after each lamp has had n-1 moves, all the lamps are off except for n-1 and 2n-1. In the next n moves lamps 1 to n-2 are turned on, lamp n-1 is turned off, lamps n to 2n-2 remain off, and lamp 2n-1 remains on. For the next n-1 moves for each lamp, lamp n-1 is not toggled, so it remains off. Hence all of n to 2n-2 also remain off and 2n-1 remains on. Lamps 0 to n-2 go through the same sequence as for a set of n lamps. Hence after these n-1 moves for each lamp, all the lamps are off, except for 2n-1. Finally, over the next 2n-1 moves, lamps 0 to 2n-2 are turned on. This completes the induction. Counting moves, we see that there are n-1 sets of n moves, followed by n-1 moves, a total of $n^2$ - 1.
(c) We show by induction on the number of moves that for n = $2^k$+ 1 lamps after each lamp has had m moves, for i = 0, 1, ... , $2^k$ - 2, lamp i+2 is in the same state as lamp i is after each lamp has had m moves in a set of n - 1 = $2^k$ lamps (we refer to this as lamp i in the *reduced* case). Lamp 0 is off and lamp 1 is on. It is easy to see that this is true for m = 1 (in both cases odd numbered lamps are on and even numbered lamps are off). Suppose it is true for m. Lamp 2 has the same state as lamp 0 in the reduced case and both toggle since their predecessor lamps are on. Hence lamps 3 to n - 1 behave the same as lamps 1 to n - 3 in the reduced case. That means that lamp n - 1 remains off. Hence lamp 0 does not toggle on its m+1th move and remains off. Hence lamp 1 does not toggle on its m+1th move and remains on. The induction stops working when lamp n - 2 toggles on its nth move in the reduced case, but it works up to and including m = n - 2. So after n - 2 moves for each lamp all lamps are off except lamp 1. In the next two moves nothing happens, then in the following n - 1 moves lamps 2 to n - 1 and lamp 0 are turned on. So all the lamps are on after a total of (n - 2)n + n + 1 = $n^2$ + n + 1 moves.

## 1994

**Problem 1**
**Solution** Take $a_1$ < $a_2$ < ... < $a_m$. Take k <= (m+1)/2. We show that $a_k$ + $a_{m-k+1}$ >= n + 1. If not, then the k distinct numbers $a_1$ + $a_{m-k+1}$, $a_2$ + $a_{m-k+1}$, ... , $a_k$ + $a_{m-k+1}$ are all <=n and hence equal to some $a_i$. But they are all greater than $a_{m-k+1}$, so each i satisfies m-k+2 <= i <= m, which is impossible since there are only k-1 available numbers in the range.

**Problem 2**
**Solution** Assume OQ is perpendicular to EF. Then angle EBO = angle EQO = 90, so EBOQ is cyclic. Hence angle OEQ = angle OBQ. Also angle OQF = angle OCF = 90, so OQCF is cyclic. Hence angle OFQ = angle OCQ. But angle OCQ = angle OBQ since ABC is isosceles. Hence angle OEQ = angle OFQ, so OE = OF, so triangles OEQ and OFQ are congruent and QE = QF.
Assume QE = QF. If OQ is not perpendicular to EF, then take E'F' through Q perpendicular to OQ with E' on AB and F' on AC. Then QE' = QF', so triangles QEE' and QFF' are congruent. Hence angle QEE' = angle QFF'. So CA and AB make the same angles with EF and hence are parallel. Contradiction. So OQ is perpendicular to EF.

**Problem 3**
**Solution** Answer: 2, 4, ... , n(n-1)/2 + 1, ... .
To get a feel, we calculate the first few values of f explicitly:
f(2) = 0, f(3) = 0, f(4) = 1 [111], f(5) = 1 [111], f(6) = 2 [111, 1011], f(7) = 3 [1011, 1101, 1110], f(8) = 3 [1011, 1101, 1110], f(9) = 3 [1011, 1101, 1110], f(10) = 4 [1011, 1101, 1110, 10011], f(11) = 5 [1101, 1110, 10011, 10101, 10110], f(12) = 5 [1101, 1110, 10011, 10101, 10110], f(13) = 6 [1110, 10011, 10101, 10110, 11001, 11010]
We show that f(k+1) = f(k) or f(k) + 1. The set for k+1 has the additional elements 2k+1 and 2k+2 and it loses the element k+1. But the binary expression for 2k+2 is the same as that for k+1 with the addition of a zero at the end, so 2k+2 and k+1 have the same number of 1s. So if 2k+1 has three 1s, then f(k+1) = f(k) + 1, otherwise f(k+1) = f(k). Now clearly an infinite number of numbers 2k+1 have three 1s, (all numbers $2^r + 2^s + 1$ for r > s > 0). So f(k) increases without limit, and since it only moves up in increments of 1, it never skips a number. In other words, given any positive integer m we can find k so that f(k) = m.
From the analysis in the last paragraph we can only have a single k with f(k) = m if both 2k-1 and 2k+1 have three 1s, or in other words if both k-1 and k have two 1s. Evidently this happens when k-1 has the form $2^n + 1$. This determines the k, namely $2^n + 2$, but we need to determine the corresponding m = f(k). It is the number of elements of $\{2^n+3, 2^n+4, ... , 2^{n+1}+4\}$ which have three 1s. Elements with three 1s are either $2^n+2^r+2^s$ with 0 <= r < s < n, or $2^{n+1}+3$. So there are m= n(n-1)/2 + 1 of them. As a check, for n = 2, we have k = $2^2+2$ = 6, m = 2, and for n = 3, we have k = $2^3+2$ = 10, m = 4, which agrees with the f(6) = 2, f(10) = 4 found earlier.

**Problem 4**
**Solution** Answer: (1, 2), (1, 3), (2, 1), (2, 2), (2, 5), (3, 1), (3, 5), (5, 2), (5, 3).
We start by checking n = 1, 2, 3. $n^3$ + 1 = 2, 9, 28. This gives the solutions (2, 1), (3, 1), (1, 2), (2, 2), (5, 2), (1, 3), (5, 3). So we assume hereafter that n > 3.
Let $n^3 + 1 = (mn - 1)h$. Then we must have h ≡ -1 (mod n). Put h = kn - 1. Then $n^3 + 1 = mkn^2 - (m + k)n + 1$. Hence $n^2 = mkn - (m + k)$. (*) Hence n divides m + k. If m + k >= 3n, then since n > 3 we have at least one of m, k >= n + 2. But then (mn - 1)(kn - 1) >= $(n^2 + 2n - 1)(n - 1) = n^3 + n^2 - 3n + 1 = (n^3 + 1) + n(n - 3) > n^3 + 1$. So we must have m + k = n or 2n.
Consider first m + k = n. We may take m >= k (provided that we remember that if m is a solution, then so is n - m). So (*) gives n = m(n - m) - 1. Clearly m = n - 1 is not a solution. If m = n - 2, then n = 2(n - 2) - 1, so n = 5. This gives the two solutions (m, n) = (2, 5) and (3, 5). If m < n - 2 then n - m >= 3 and so m(n - m) - 1 >= 3m - 1 >= 3n/2 - 1 > n for n > 3.
Finally, take m + k = 2n. So (*) gives n + 2 = m(2n - m). Again we may take m >= k. m = 2n - 1 is not a solution (we are assuming n > 3). So 2n - m >= 2, and hence m(2n - m) >= 2m >= 2n > n + 2.

**Problem 5**
**Solution** Answer: f(x) = -x/(x+1)
Suppose f(a) = a. Then putting x = y = a in the relation given, we get f(b) = b, where b = $2a + a^2$. If -1 < a < 0, then -1 < b < a. But f(a)/a = f(b)/b. Contradiction. Similarly, if a > 0, then b > a, but f(a)/a = f(b)/b. Contradiction. So we must have a = 0.
But putting x = y in the relation given we get f(a) = a for a = x + f(x) + xf(x). Hence for any x we have x + f(x) + xf(x) = 0 and hence f(x) = -x/(x+1).





Finally, it is straightforward to check that f(x) = -x(x+1) satisfies the two conditions.
Thanks to Gerhard Woeginger for pointing out the error in the original solution and supplying this solution.

**Problem 6**
**Solution** Let the primes be $p_1 < p_2 < p_3 < ...$ . Let A consists of all products of n distinct primes such that the smallest is greater than $p_n$. For example: all primes except 2 are in A; 21 is not in A because it is a product of two distinct primes and the smallest is greater than 3. Now let S = $\{p_{i1}, p_{i2}, ... \}$ be any infinite set of primes. Assume that $p_{i1} < p_{i2} < ...$ . Let n = $i_1$. Then $p_{i1}p_{i2} ... p_{in}$ is not in A because it is a product of n distinct primes, but the smallest is not greater than $p_n$. But $p_{i2}p_{i3} ... p_{in+1}$ is in A, because it is a product of n distinct primes and the smallest is greater than $p_n$. But both numbers are products of n distinct elements of S.

## 1995

**Problem 1**
**Solution** Let DN meet XY at Q. Angle QDZ = 90 - angle NBD = angle BPZ. So triangles QDZ and BPZ are similar. Hence QZ/DZ = BZ/PZ, or QZ = BZ.DZ/PZ. Let AM meet XY at Q'. Then the same argument shows that Q'Z = AZ.CZ/PZ. But BZ.DZ = XZ.YZ = AZ.CZ, so QZ = Q'Z. Hence Q and Q' coincide.

**Problem 2**
**Solution** Put a = 1/x, b = 1/y, c = 1/z. Then $1/(a^3(b+c)) = x^3yz/(y+z) = x^2/(y+z)$. Let the expression given be E. Then by Cauchy's inequality we have (y+z + z+x + x+y)E >= $(x + y + z)^2$, so E >= (x + y + z)/2. But applying the arithmetic/geometric mean result to x, y, z gives (x + y + z) >= 3. Hence result.
Thanks to Gerhard Woeginger for pointing out that the original solution was wrong.

**Problem 3**
**Solution** The first point to notice is that if no arrangement is possible for n, then no arrangement is possible for any higher integer. Clearly the four points of a square work for n = 4, so we focus on n = 5.
If the 5 points form a convex pentagon, then considering the quadrilateral $A_1A_2A_3A_4$ as made up of two triangles in two ways, we have that $r_1 + r_3 = r_2 + r_4$. Similarly, $A_5A_1A_2A_3$ gives $r_1 + r_3 = r_2 + r_5$, so $r_4 = r_5$.
We show that we cannot have two r's equal (whether or not the 4 points form a convex pentagon). For suppose $r_4 = r_5$. Then $A_1A_2A_4$ and $A_1A_2A_5$ have equal area. If $A_4$ and $A_5$ are on the same side of the line $A_1A_2$, then since they must be equal distances from it, $A_4A_5$ is parallel to $A_1A_2$. If they are on opposite sides, then the midpoint of $A_4A_5$ must lie on $A_1A_2$. The same argument can be applied to $A_1$ and $A_3$, and to $A_2$ and $A_3$. But we cannot have two of $A_1A_2$, $A_1A_3$ and $A_2A_3$ parallel to $A_4A_5$, because then $A_1$, $A_2$ and $A_3$ would be collinear. We also cannot have the midpoint of $A_4A_5$ lying on two of $A_1A_2$, $A_1A_3$ and $A_2A_3$ for the same reason. So we have established a contradiction. hence no two of the r's can be equal. In particular, this shows that the 5 points cannot form a convex pentagon.
Suppose the convex hull is a quadrilateral. Without loss of generality, we may take it to be $A_1A_2A_3A_4$. $A_5$ must lie inside one of $A_1A_2A_4$ and $A_2A_3A_4$. Again without loss of generality we may take it to be the latter, so that $A_1A_2A_5A_4$ is also a convex

quadrilateral. Then $r_2 + r_4 = r_1 + r_3$ and also $= r_1 + r_5$. So $r_3 = r_5$, giving a contradiction as before.
The final case is the convex hull a triangle, which we may suppose to be $A_1A_2A_3$. Each of the other two points divides its area into three triangles, so we have: $(r_1 + r_2 + r_4) + (r_2 + r_3 + r_4) + (r_3 + r_1 + r_4) = (r_1 + r_2 + r_5) + (r_2 + r_3 + r_5) + (r_3 + r_1 + r_5)$ and hence $r_4 = r_5$, giving a contradiction.
So the arrangement is not possible for 5 and hence not for any $n > 5$.

**Problem 4**
**Solution** Answer: $2^{997}$.
The relation given is a quadratic in $x_i$, so it has two solutions, and by inspection these are $x_i = 1/x_{i-1}$ and $x_i/2$. For an even number of moves we can start with an arbitary $x_0$ and get back to it. Use n-1 halvings, then take the inverse, that gets to $2^{n-1}/x_0$ after n moves. Repeating brings you back to $x_0$ after 2n moves. However, 1995 is odd!
The sequence given above brings us back to $x_0$ after n moves, provided that $x_0 = 2^{(n-1)/2}$. We show that this is the largest possible $x_0$. Suppose we have a halvings followed by an inverse followed by b halvings followed by an inverse. Then if the number of inverses is odd we end up with $2^{a-b+c- \dots}/x_0$, and if it is even we end up with $x_0/2^{a-b+c- \dots}$. In the first case, since the final number is $x_0$ we must have $x_0 = 2^{(a-b+-\dots)/2}$. All the a, b, ... are non-negative and sum to the number of moves less the number of inverses, so we clearly maximise $x_0$ by taking a single inverse and a = n-1. In the second case, we must have $2^{a-b+c- \dots} = 1$ and hence a - b + c - ... = 0. But that implies that a + b + c + ... is even and hence the total number of moves is even, which it is not. So we must have an odd number of inverses and the maximum value of $x_0$ is $2^{(n-1)/2}$.

**Problem 5**
**Solution** BCD is an equilateral triangle and AEF is an equilateral triangle. The presence of equilateral triangles and quadrilaterals suggests using Ptolemy's inequality. From CBGD, we get CG.BD >= BG.CD + GD.CB, so CG >= BG + GD. Similarly from HAFE we get HF >= HA + HE. Then CF >= CG + GH + HF. But we do not get quite what we want. However, BAED is symmetrical about BE (because BA = BD and EA = ED). So we may take C' the reflection of C in the line BE and F' the reflection of F. Now C'AB and F'ED are still equilateral, so the same argument gives C'F >= AG + GB and HF' >= DH + HE. We now have the desired inequality.

**Problem 6**
**Solution** Answer: 2 + (2pCp - 2)/p, where 2pCp is the binomial coefficient (2p)!/(p!p!).
Let A be a subset other than {1, 2, ... , p} and {p+1, p+2, ... , 2p}. Consider the elements of A in {1, 2, ... , p}. The number r satisfies 0 < r < p. We can change these elements to another set of r elements of {1, 2, ... , p} by adding 1 to each element (and reducing mod p if necessary). We can repeat this process and get p sets in all. For example, if p = 7 and the original subset of {1, 2, ... , 7} was {3 , 5}, we get:
{3 , 5}, {4, 6}, {5, 7}, {6, 1}, {7, 2}, {1, 3}, {2, 4}.
The sum of the elements in the set is increased by r each time. So, since p is prime, the sums must form a complete set of residues mod p. In particular, they must all be distinct and hence all the subsets must be different.
Now consider the sets A which have a given intersection with {p+1, ... , n}. Suppose the elements in this intersection sum to k mod p. The sets can be partitioned into groups of p by the process described above, so that exactly one member of each group will have





the sum -k mod p for its elements in {1, 2, ... , p}. In other words, exactly one member of each group will have the sum of all its elements divisible by p.
There are 2pCp subsets of {1, 2, ... , 2p} of size p. Excluding {1, 2, ... , p} and {p+1, ... , 2p} leaves (2pCp - 2). We have just shown that (2pCp - 2)/p of these have sum divisible by p. The two excluded subsets also have sum divisible by p, so there are 2 + (2pCp - 2)/p subsets in all having sum divisible by p.

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## 1996

**Problem 1**
**Solution** (a) Suppose the move is a units in one direction and b in the orthogonal direction. So $a^2 + b^2 = r$. If r is divisible by 2, then a and b are both even or both odd. But that means that we can only access the black squares or the white squares (assuming the rectangle is colored like a chessboard). The two corners are of opposite color, so the task cannot be done. All squares are congruent to 0 or 1 mod 3, so if r is divisible by 3, then a and b must both be multiples of 3. That means that if the starting square has coordinates (0,0), we can only move to squares of the form (3m,3n). The required destination is (19,0) which is not of this form, so the task cannot be done.
(b) If r = 73, then we must have a = 8, b = 3 (or vice versa). There are 4 types of move:
A: (x,y) to (x+8,y+3)
B: (x,y) to (x+3,y+8)
C: (x,y) to (x+8,y-3)
D: (x,y) to (x+3,y-8)
We regard (x,y) to (x-8,y-3) as a negative move of type A, and so on. Then if we have a moves of type A, b of type B and so on, then we require:
$8(a + c) + 3(b + d) = 19$; $3(a - c) + 8(b - d) = 0$.
A simple solution is a = 5, b = -1, c = -3, d = 2, so we start by looking for solutions of this type. After some fiddling we find:
(0,0) to (8,3) to (16,6) to (8,9) to (11,1) to (19,4) to (11,7) to (19,10) to (16,2) to (8,5) to (16,8) to (19,0).
(c) If r = 97, then we must have a = 9, b = 4. As before, assume we start at (0,0). A good deal of fiddling around fails to find a solution, so we look for reasons why one is impossible. Call moves which change y by 4 "toggle" moves. Consider the central strip y = 4, 5, 6 or 7. Toggle moves must toggle us in and out of the strip. Non-toggle moves cannot be made if we are in the strip and keep us out of it if we are out of it. Toggle moves also change the parity of the x-coordinate, whereas non-toggle moves do not. Now we start and finish out of the strip, so we need an even number of toggle moves. On the other hand, we start with even x and end with odd x, so we need an odd number of toggle moves. Hence the task is impossible.

**Problem 2**
**Solution** We need two general results: the angle bisector theorem; and the result about the feet of the perpendiculars from a general point inside a triangle. The second is not so well-known. Let P be a general point in the triangle ABC with X, Y, Z the feet of the perpendiculars to BC, CA, AB. Then PA = YZ/sin A and angle APB - angle C = angle XZY. To prove the first part: AP = AY/sin APY = AY/sin AZY (since AYPZ is cyclic) = YZ/sin A (sine rule). To prove the second part: angle XZY = angle XZP + angle YZP = angle XBP + angle YAP = 90 - angle XPB + 90 - angle YPA = 180 - (360 - angle APB - angle XPY) = -180 + angle APB + (180 - angle C) = angle APB - angle C.

So, returning to the problem, angle APB - angle C = angle XZY and angle APC - angle B = angle XYZ. Hence XYZ is isosceles: XY = XZ. Hence PC sin C = PB sin B. But AC sin C = AB sin B, so AB/PB = AC/PC. Let the angle bisector BD meet AP at W. Then, by the angle bisector theorem, AB/PB = AW/WP. Hence AW/WP = AC/PC, so, by the angle bisector theorem, CW is the bisector of angle ACP, as required.

**Problem 3**

**Solution** Setting m = n = 0, the given relation becomes: f(f(0)) = f(f(0)) + f(0). Hence f(0) = 0. Hence also f(f(0)) = 0. Setting m = 0, now gives f(f(n)) = f(n), so we may write the original relation as f(m + f(n)) = f(m) + f(n).
So f(n) is a fixed point. Let k be the smallest non-zero fixed point. If k does not exist, then f(n) is zero for all n, which is a possible solution. If k does exist, then an easy induction shows that f(qk) = qk for all non-negative integers q. Now if n is another fixed point, write n = kq + r, with 0 <= r < k. Then f(n) = f(r + f(kq)) = f(r) + f(kq) = kq + f(r). Hence f(r) = r, so r must be zero. Hence the fixed points are precisely the multiples of k. But f(n) is a fixed point for any n, so f(n) is a multiple of k for any n. Let us take $n_1$, $n_2$, ... , $n_{k-1}$ to be arbitary non-negative integers and set $n_0$ = 0. Then the most general function satisfying the conditions we have established so far is:
f(qk + r) = qk + $n_r$k for 0 <= r < k.
We can check that this satisfies the functional equation. Let m = ak + r, n = bk + s, with 0 <= r, s < k. Then f(f(m)) = f(m) = ak + $n_r$k, and f(n) = bk + $n_s$k, so f(m + f(n)) = ak + bk + $n_r$k + $n_s$k, and f(f(m)) + f(n) = ak + bk + $n_r$k + $n_s$k. So this is a solution and hence the most general solution.

**Problem 4**

**Solution** Answer: $481^2$.
Put 15a -+ 16b = $m^2$, 16a - 15b = $n^2$. Then $15m^2 + 16n^2$ = 481a = 13.37a. The quadratic residues mod 13 are 0, +-1, +-3, +-4, so the residues of $15m^2$ are 0, +-2, +-5, +-6, and the residues of $16n^2$ are 0, +-1, +-3, +-4. Hence m and n must both be divisible by 13. Similarly, the quadratic residues of 37 are 0, +-1, +-3, +-4, +-7, +-9, +-10, +-11, +-12, +-16, so the residues of $15m^2$ are 0, +-2, +-5, +-6, +-8, +-13, +-14, +-15, +-17, +-18, and the residues of $16n^2$ are 0, +-1, +-3, +-4, +-7, +-9, +-10, +-11, +-12, +-16. Hence m and n must both be divisible by 37. Put m = 481m', n = 481n' and we get: a = 481(15$m'^2$ + 16$n'^2$. We also have 481b = $16m^2 - 15n^2$ and hence b = 481(16$m'^2$ - 15$n'^2$). The smallest possible solution would come from putting m' = n' = 1 and indeed that gives a solution.
This solution is straightforward, but something of a slog - all the residues have to be calculated. A more elegant variant is to notice that $m^4 + n^4 = 481(a^2 + b^2)$. Now if m and n are not divisible by 13 we have $m^4 + n^4 \equiv 0$ (mod 13). Take k so that km ≡ 1 (mod 13), then $(nk)^4 \equiv -(mk)^4 \equiv -1$ (mod 13). But that is impossible because then $(nk)^{12} \equiv -1$ (mod 13), but $x^{12} \equiv 1$ (mod 13) for all non-zero residues. Hence m and n are both divisible by 13. The same argument shows that m and n are both divisible by 37.

**Problem 5**

**Solution** The starting point is the formula for the circumradius R of a triangle ABC: 2R = a/sin A = b/sin B = c/sin C. [Proof: the side a subtends an angle 2A at the center, so a = 2R sin A.] This gives that $2R_A$ = BF/sin A, $2R_C$ = BD/sin C, $2R_E$ = FD/sin E. It is clearly not true in general that BF/sin A > BA + AF, although it is true if angle FAB >= 120, so we need some argument that involves the hexagon as a whole.





Extend sides BC and FE and take lines perpendicular to them through A and D, thus forming a rectangle. Then BF is greater than or equal to the side through A and the side through D. We may find the length of the side through A by taking the projections of BA and AF giving AB sin B + AF sin F. Similarly the side through D is CD sin C + DE sin E. Hence:
2BF >= AB sin B + AF sin F + CD sin C + DE sin E. Similarly:
2BD >= BC sin B + CD sin D + AF sin A + EF sin E, and
2FD >= AB sin A + BC sin C + DE sin D + EF sin F.
Hence 2BF/sin A + 2BD/sin C + 2FD/sin E >= AB(sin A/sin E + sin B/sin A) + BC(sin B/sin C + sin C/sin E) + CD(sin C/sin A + sin D/sin C) + DE(sin E/sin A + sin D/sin E) + EF(sin E/sin C + sin F/sin E) + AF(sin F/sin A + sin A/sin C).
We now use the fact that opposite sides are parallel, which implies that opposite angles are equal: A = E, B = E, C = F. Each of the factors multiplying the sides in the last expression now has the form x + 1/x which has minimum value 2 when x = 1. Hence 2(BF/sin A + BD/sin C + FD/sin E) >= 2p and the result is proved.

**Problem 6**

**Solution** Let $x_i - x_{i-1}$ = p occur r times and $x_i - x_{i-1}$ = -q occur s times. Then r + s = n and pr = qs. If p and q have a common factor d, the $y_i = x_i/d$ form a similar set with p/d and q/d. If the result is true for the $y_i$ then it must also be true for the $x_i$. So we can assume that p and q are relatively prime. Hence p divides s. Let s = kp. If k = 1, then p = s and q = r, so p + q = r + s = n. But we are given p + q < n. Hence k > 1. Let p + q = n/k = h. Up to this point everything if fairly obvious and the result looks as though it should be easy, but I did not find it so. Some fiddling around with examples suggested that we seem to get $x_i = x_j$ for j = i + h. We observe first that $x_{i+h} - x_i$ must be a multiple of h. For suppose e differences are p, and hence h-e are -q. Then $x_{i+h} - x_i$ = ep - (h - e)q = (e - q)h.
The next step is not obvious. Let $d_i = x_{i+h} - x_i$. We know that all $d_i$s are multiples of h. We wish to show that at least one is zero. Now $d_{i+1} - d_i = (x_{i+h+1} - x_{i+h}) - (x_{i+1} - x_i)$ = (p or -q) - (p or -q) = 0, h or -h. So if neither of $d_i$ nor $d_{i+1}$ are zero, then either both are positive or both are negative (a jump from positive to negative would require a difference of at least 2h). Hence if none of the $d_i$s are zero, then all of them are positive, or all of them are negative. But $d_0 + d_h + ... + d_{kh}$ is a concertina sum with value $x_n - x_0$ = 0. So this subset of the $d_i$s cannot all be positive or all negative. Hence at least one $d_i$ is zero.

## 1997

**Problem 1**

**Solution** (a) If m and n are both even, then f(m,n) = 0. Let M be the midpoint of the hypoteneuse. The critical point is that M is a lattice point. If we rotate the triangle through 180 to give the other half of the rectangle, we find that its coloring is the same. Hence $S_1$ and $S_2$ for the triangle are each half their values for the rectangle. But the values for the rectangle are equal, so they must also be equal for the triangle and hence f(m,n) = 0.
If m and n are both odd, then the midpoint of the hypoteneuse is the center of a square and we may still find that the coloring of the two halves of the rectangle is the same. This time $S_1$ and $S_2$ differ by one for the rectangle, so f(m,n) = 1/2.

(b) The result is immediate from (a) for m and n of the same parity. The argument in (a) fails for m and n with opposite parity, because the two halves of the rectangle are oppositely colored. Let m be the odd side. Then if we extend the side length m by 1 we form a new triangle which contains the original triangle. But it has both sides even and hence $S_1 = S_2$. The area added is a triangle base 1 and height n, so area n/2. The worst case would be that all this area was the same color, in which case we would get f(m,n) = n/2. But n <= max(m,n), so this establishes the result.
(c) Intuitively, it is clear that if the hypoteneuse runs along the diagonal of a series of black squares, and we then extend one side, the extra area taken in will be mainly black. We need to make this rigorous. For the diagonal to run along the diagonal of black squares we must have n = m. It is easier to work out the white area added by extending a side. The white area takes the form of a series of triangles each similar to the new n+1 x n triangle. The biggest has sides 1 and n/(n+1). The next biggest has sides (n-1)/n and (n-1)/(n+1), the next biggest (n-2)/n and (n-2)/(n+1) and so on, down to the smallest which is 1/n by 1/(n+1). Hence the additional white area is $1/2\ (n/(n+1) + (n-1)^2/(n(n+1)) + (n-2)^2/(n(n+1)) + \dots + 1/(n(n+1))\ ) = 1/(2n(n+1))\ \ (n^2 + \dots + 1^2) = (2n+1)/12$. Hence the additional black area is n/2 - (2n+1)/12 = n/3 - 1/12 and the black excess in the additional area is n/6 - 1/6. If n is even, then f(n,n) = 0 for the original area, so for the new triangle f(n+1,n) = (n-1)/6 which is unbounded.

**Problem 2**
**Solution** Extend BV to meet the circle again at X, and extend CW to meet the circle again at Y. Then by symmetry (since the perpendicular bisectors pass through the center of the circle) AU = BX and AU = CY. Also arc AX = arc BU, and arc AY = arc UC. Hence arc XY = arc BC and so angle BYC = angle XBY and hence TY = TB. So AU = CY = CT + TY = CT + TB.

**Problem 3**
**Solution** Without loss of generality we may assume $x_1 + \dots + x_n = +1$. [If not just reverse the sign of every $x_i$.] For any given arrangement $x_i$ we use *sum* to mean $x_1 + 2x_2 + 3x_3 + \dots + nx_n$. Now if we add together the sums for $x_1, x_2, \dots, x_n$ and the *reverse* $x_n, x_{n-1}, \dots, x_1$, we get $(n+1)(x_1 + \dots + x_n) = n+1$. So either we are home with the original arrangement or its reverse, or they have sums of opposite sign, one greater than (n+1)/2 and one less than -(n+1)/2.
A transposition changes the sum from ka + (k+1)b + other terms to kb + (k+1)a + other terms. Hence it changes the sum by |a - b| (where a, b are two of the $x_i$) which does not exceed n+1. Now we can get from the original arrangement to its reverse by a sequence of transpositions. Hence at some point in this sequence the sum must fall in the interval [-(n+1)/2, (n+1)/2] (because to get from a point below it to a point above it in a single step requires a jump of more than n+1). That point gives us the required permutation.

**Problem 4**
**Solution** (a) If we list all the elements in the rows followed by all the elements in the columns, then we have listed every element in the array twice, so each number in S must appear an even number of times. But considering the ith row with the ith column, we have also given n complete copies of S together with an additional copy of the numbers on the diagonal. If n is odd, then each of the 2n-1 numbers appears an odd number of times in the n complete copies, and at most n numbers can have this converted to an even number by an appearance on the diagonal. So there are no silver matrices for n odd. In particular, there is no silver matrix for n = 1997.





(b) Let $A_{i,j}$ be an n x n silver matrix with 1s down the main diagonal. Define the 2n x 2n matrix $B_{i,j}$ with 1s down the main diagonal as follows: $B_{i,j} = A_{i,j}$; $B_{i+n,j+n} = A_{i,j}$; $B_{i,j+n} = 2n + A_{i,j}$; $B_{i+n,j} = 2n + A_{i,j}$ for i not equal j and $B_{i+n,i} = 2n$. We show that $B_{i,j}$ is silver. Suppose i <= n. Then the first half of the ith row is the ith row of $A_{i,j}$, and the top half of the ith column is the ith column of $A_{i,j}$, so between them those two parts comprise the numbers from 1 to 2n - 1. The second half of the ith row is the ith row of $A_{i,j}$ with each element increased by 2n, and the bottom half of the ith column is the ith column of $A_{i,j}$ with each element increased by 2n, so between them they give the numbers from 2n + 1 to 4n - 1. The only exception is that $A_{i+n,i} = 2n$ instead of $2n + A_{i,i}$. We still get $2n + A_{i,i}$ because it was in the second half of the ith row (these two parts do not have an element in common). The 2n fills the gap so that in all we get all the numbers from 1 to 4n - 1. An exactly similar argument works for i > n. This time the second half of the row and the second half of the column (which overlap by one element) give us the numbers from 1 to 2n - 1, and the first halves (which do not overlap) give us 2n to 4n - 1. So $B_{i,j}$ is silver. Hence there are an infinite number of silver matrices.

**Problem 5**
**Solution** Answer: (1,1), (16,2), (27,3).
Notice first that if we have $a^m = b^n$, then we must have $a = c^e$, $b = c^f$, for some c, where m=ed, n=fd and d is the greatest common divisor of m and n. [Proof: express a and b as products of primes in the usual way.]
In this case let d be the greatest common divisor of a and $b^2$, and put a = de, $b^2$ = df. Then for some c, $a = c^e$, $b = c^f$. Hence $f\ c^e = e\ c^{2f}$. We cannot have e = 2f, for then the c's cancel to give e = f. Contradiction. Suppose 2f > e, then $f = e\ c^{2f-e}$. Hence e = 1 and $f = c^{2f-1}$. If c = 1, then f = 1 and we have the solution a = b = 1. If c >= 2, then $c^{2f-1} >= 2^f > f$, so there are no solutions.
Finally, suppose 2f < e. Then $e = f\ c^{e-2f}$. Hence f = 1 and $e = c^{e-2}$. $c^{e-2} >= 2^{e-2} >= e$ for e >= 5, so we must have e = 3 or 4 (e > 2f = 2). e = 3 gives the solution a = 27, b = 3. e = 4 gives the solution a = 16, b = 2.

**Problem 6**
**Solution** The key is to derive a recurrence relation for f(n) [not for $f(2^n)$]. If n is odd, then the sum must have a 1. In fact, there is a one-to-one correspondence between sums for n and sums for n-1. So:

f(2n+1) = f(2n)

Now consider n even. The same argument shows that there is a one-to-one correspondence between sums for n-1 and sums for n which have a 1. Sums which do not have a 1 are in one-to-one correspondence with sums for n/2 (just halve each term). So:

f(2n) = f(2n - 1) + f(n) = f(2n - 2) + f(n).

The upper limit is now almost immediate. First, the recurrence relations show that f is monotonic increasing. Now apply the second relation repeatedly to $f(2^{n+1})$ to get:

$f(2^{n+1}) = f(2^{n+1} - 2^n) + f(2^n - 2^{n-1} + 1) + \dots + f(2^n - 1) + f(2^n) = f(2^n) + f(2^n - 1) + \dots + f(2^{n-1} + 1) + f(2^n)$ (*)

and hence $f(2^{n+1}) >= (2^{n-1} + 1)f(2^n)$
We can now establish the upper limit by induction. It is false for n = 1 and 2, but almost true for n = 2, in that: $f(2^2) = 2^{22/2}$. Now if $f(2^n) <= 2^{n2/2}$, then the inequality just established shows that $f(2^{n+1}) < 2^n 2^{n2/2} < 2^{(n2+2n+1)/2} = 2^{(n+1)2/2}$, so it is true for n + 1. Hence it is true for all n > 2.

Applying the same idea to the lower limit does not work. We need something stronger. We may continue (*) inductively to obtain $f(2^{n+1}) = f(2^n) + f(2^n - 1) + ... + f(3) + f(2) + f(1) + 1$. (**) We now use the following lemma:

$f(1) + f(2) + ... + f(2r) \geq 2r\, f(r)$

We group the terms on the lhs into pairs and claim that $f(1) + f(2r) \geq f(2) + f(2r-1) \geq f(3) + f(2r-2) \geq ... \geq f(r) + f(r+1)$. If k is even, then $f(k) = f(k+1)$ and $f(2r-k) = f(2r+1-k)$, so $f(k) + f(2r+1-k) = f(k+1) + f(2r-k)$. If k is odd, then $f(k+1) = f(k) + f((k+1)/2)$ and $f(2r+1-k) = f(2r-k) + f((2r-k+1)/2)$, but f is monotone so $f((k+1)/2) \leq f((2r+1-k)/2)$ and hence $f(k) + f(2r+1-k) \geq f(k+1) + f(2r-k)$, as required.

Applying the lemma to (**) gives $f(2^{n+1)} > 2^{n+1}f(2^{n-1})$. This is sufficient to prove the lower limit by induction. It is true for n = 1. Suppose it is true for n. Then $f(2^{n+1}) > 2^{n+1}2^{(n-1)2/4} = 2^{(n2-2n+1+4n+4)/4} > 2^{(n+1)2/4}$, so it is true for n+1.

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## 1998

**Problem 1**

**Solution** Let AC and BD meet at X. Let H, K be the feet of the perpendiculars from P to AC, BD respectively. We wish to express the areas of ABP and CDP in terms of more tractable triangles. There are essentially two different configurations possible. In the first, we have area PAB = area ABX + area PAX + area PBX, and area PCD = area CDX - area PCX - area PDX. So if the areas being equal is equivalent to: area ABX - area CDX + area PAX + area PCX + area PBX + area PDX. ABX and CDX are right-angled, so we may write their areas as AX.BX/2 and CX.DX/2. We may also put AX = AM - MX = AM - PN, BX = BN - PM, CX = CM + PN, DX = DN + PM. The other triangles combine in pairs to give area ACP + area BDP = AC.PM + BD.PN. This leads, after some cancellation to AM.BN = CM.DN. There is a similar configuration with the roles of AB and CD reversed.

The second configuration is area PAB = area ABX + area PAX - PBX, area PCD = area CDX + area PDX - area PCX. In this case AX = AM + PN, BX = BN - PM, CX = CM - PN, DX = DN + PM. But we end up with the same result: AM.BN = CM.DN.

Now if ABCD is cyclic, then it follows immediately that P is the center of the circumcircle and AM = CM, BN = DN. Hence the areas of PAB and PCD are equal. Conversely, suppose the areas are equal. If PA > PC, then AM > CM. But since PA = PB and PC = PD (by construction), PB > PD, so BN > DN. So AM.BN > CM.DN. Contradiction. So PA is not greater than PC. Similarly it cannot be less. Hence PA = PC. But that implies PA = PB = PC = PD, so ABCD is cyclic.

**Problem 2**

**Solution** Let us count the number N of triples (judge, judge, contestant) for which the two judges are distinct and rate the contestant the same. There are $b(b-1)/2$ pairs of judges in total and each pair rates at most k contestants the same, so $N \leq kb(b-1)/2$. Now consider a fixed contestant X and count the number of pairs of judges rating X the same. Suppose x judges pass X, then there are $x(x-1)/2$ pairs who pass X and $(b-x)(b-x-1)/2$ who fail X, so a total of $(x(x-1) + (b-x)(b-x-1))/2$ pairs rate X the same. But $(x(x-1) + (b-x)(b-x-1))/2 = (2x^2 - 2bx + b^2 - b)/2 = (x - b/2)^2 + b^2/4 - b/2 \geq b^2/4 - b/2 = (b - 1)^2/4 - 1/4$. But $(b - 1)^2/4$ is an integer (since b is odd), so the number of pairs rating X the same is at least $(b - 1)^2/4$. Hence $N \geq a\,(b - 1)^2/4$. Putting the two inequalities together gives $k/a \geq (b - 1)/2b$.





**Problem 3**

**Solution** Let $n = p_1{}^a{}_1...p_r{}^a{}_r$. Then $d(n) = (a_1 + 1)(a_2 + 1) ... (a_r + 1)$, and $d(n^2) = (2a_1 + 1)(2a_2 + 1) ... (2a_r + 1)$. So the $a_i$ must be chosen so that $(2a_1 + 1)(2a_2 + 1) ... (2a_r + 1) = k\,(a_1 + 1)(a_2 + 1) ... (a_r + 1)$. Since all $(2a_i + 1)$ are odd, this clearly implies that k must be odd. We show that conversely, given any odd k, we can find $a_i$.

We use a form of induction on k. First, it is true for k = 1 (take n = 1). Second, we show that if it is true for k, then it is true for $2^mk - 1$. That is sufficient, since any odd number has the form $2^mk - 1$ for some smaller odd number k. Take $a_i = 2^i((2^m - 1)k - 1)$ for i = 0, 1, ... , m-1. Then $2a_i + 1 = 2^{i+1}(2^m - 1)k - (2^{i+1} - 1)$ and $a_i + 1 = 2^i(2^m - 1)k - (2^i - 1)$. So the product of the $(2a_i + 1)$'s divided by the product of the $(a_i + 1)$'s is $2^m(2^m - 1)k - (2^m - 1)$ divided by $(2^m - 1)k$, or $(2^mk - 1)/k$. Thus if we take these $a_i$s together with those giving k, we get $2^mk - 1$, which completes the induction.

**Problem 4**

**Solution** Answer: (a, b) = (11, 1), (49, 1) or $(7k^2, 7k)$.

If a < b, then $b \geq a + 1$, so $ab^2 + b + 7 > ab^2 + b \geq (a + 1)(ab + 1) = a^2b + a + ab \geq a^2b + a + b$. So there can be no solutions with a < b. Assume then that $a \geq b$.

Let k = the integer $(a^2b + a + b)/(ab^2 + b + 7)$. We have $(a/b + 1/b)(ab^2 + b + 7) = ab^2 + a + ab + 7a/b + 7/b + 1 > ab^2 + a + b$. So $k < a/b + 1/b$. Now if $b \geq 3$, then $(b - 7/b) > 0$ and hence $(a/b - 1/b)(ab^2 + b + 7) = ab^2 + a - a(b - 7/b) - 1 - 7/b < ab^2 + a < ab^2 + a + b$. Hence either b = 1 or 2 or $k > a/b - 1/b$.

If $a/b - 1/b < k < a/b + 1/b$, then $a - 1 < kb < a + 1$. Hence a = kb. This gives the solution $(a, b) = (7k^2, 7k)$.

It remains to consider b = 1 and 2. If b = 1, then a + 8 divides $a^2 + a + 1$ and hence also $a(a + 8) - (a^2 + a + 1) = 7a - 1$, and hence also $7(a + 8) - (7a - 1) = 57$. The only factors bigger than 8 are 19 and 57, so a = 11 or 49. It is easy to check that (a, b) = (11, 1) and (49, 1) are indeed solutions.

If b = 2, then 4a + 9 divides $2a^2 + a + 2$, and hence also $a(4a + 9) - 2(2a^2 + a + 2) = 7a - 4$, and hence also $7(4a + 9) - 4(7a - 4) = 79$. The only factor greater than 9 is 79, but that gives a = 35/2 which is not integral. Hence there are no solutions for b = 2.

**Problem 5**

**Solution** We show that $RI^2 + SI^2 - RS^2 > 0$. The result then follows from the cosine rule. BI is perpendicular to MK and hence also to RS. So $IR^2 = BR^2 + BI^2$ and $IS^2 = BI^2 + BS^2$. Obviously RS = RB + BS, so $RS^2 = BR^2 + BS^2 + 2\,BR.BS$. Hence $RI^2 + SI^2 - RS^2 = 2\,BI^2 - 2\,BR.BS$. Consider the triangle BRS. The angles at B and M are 90 - B/2 and 90 - A/2, so the angle at R is 90 - C/2. Hence BR/BM = cos A/2/cos C/2 (using the sine rule). Similarly, considering the triangle BKS, BS/BK = cos C/2/cos A/2. So $BR.BS = BM.BK = BK^2$. Hence $RI^2 + SI^2 - RS^2 = 2(BI^2 - BK^2) = 2\,IK^2 > 0$.

**Problem 6**

**Solution** Answer: 120.

Let f(1) = k. Then $f(kt^2) = f(t)^2$ and $f(f(t)) = k^2t$. Also $f(kt)^2 = 1.f(kt)^2 = f(k^3t^2) = f(1^2f(f(kt^2))) = k^2f(kt^2) = k^2f(t)^2$. Hence f(kt) = kf(t).

By an easy induction $k^nf(t^{n+1}) = f(t)^{n+1}$. But this implies that k divides f(t). For suppose the highest power of a prime p dividing k is a > b, the highest power of p dividing f(t). Then a > b(1 + 1/n) for some integer n. But then na > (n + 1)b, so $k^n$ does not divide $f(t)^{n+1}$. Contradiction.

Let g(t) = f(t)/k. Then $f(t^2f(s)) = f(t^2kg(s)) = kf(t^2g(s) = k^2g(t^2g(s))$, whilst $s\,f(t)^2 = k^2s\,f(t)^2$. So $g(t^2g(s)) = s\,g(t)^2$. Hence g is also a function satisfying the conditions which

evidently has smaller values than f (for $k > 1$). It also satisfies $g(1) = 1$. Since we want the smallest possible value of f(1998) we may restrict attention to functions f satisfying $f(1) = 1$.
Thus we have $f(f(t) = t$ and $f(t^2) = f(t)^2$. Hence $f(st)^2 = f(s^2t^2) = f(s^2f(f(t^2))) = f(s)^2f(t^2) = f(s)^2f(t)^2$. So $f(st) = f(s) f(t)$.
Suppose p is a prime and $f(p) = m.n$. Then $f(m)f(n) = f(mn) = f(f(p)) = p$, so one of f(m), $f(n) = 1$. But if $f(m) = 1$, then $m = f(f(m)) = f(1) = 1$. So f(p) is prime. If $f(p) = q$, then $f(q) = p$.
Now we may define f arbitarily on the primes subject only to the conditions that each f(prime) is prime and that if $f(p) = q$, then $f(q) = p$. For suppose that $s = p_1^{a_1}...p_r^{a_r}$ and that $f(p_i) = q_i$. If t has any additional prime factors not included in the $q_i$, then we may add additional p's to the expression for s so that they are included (taking the additional a's to be zero). So suppose $t = q_1^{b_1}...q_r^{b_r}$.Then $t^2f(s) = q_1^{2b_1+a_1} ...q_r^{2b_r+a_r}$ and hence $f(t^2f(s) = p_1^{2b_1+a_1} ...p_r^{2b_r+a_r} = s f(t)^2$.
We want the minimum possible value of f(1998). Now $1998 = 2.3^3.37$, so we achieve the minimum value by taking $f(2) = 3$, $f(3) = 2$, $f(37) = 5$ (and $f(37) = 5$). This gives $f(1998) = 3.2^3.5 = 120$.

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## 1999

**Problem 1**
**Solution**
The possible sets are just the regular n-gons ($n > 2$).
Let $A_1, A_2, ... , A_k$ denote the vertices of the convex hull of S (and take indices mod k as necessary). We show first that these form a regular k-gon. $A_{i+1}$ must lie on the perpendicular bisector of $A_i$ and $A_{i+2}$ (otherwise its reflection would lie outside the hull). Hence the sides are all equal. Similarly, $A_{i+1}$ and $A_{i+2}$ must be reflections in the perpendicular bisector of $A_i$ and $A_{i+3}$ (otherwise one of the reflections would lie outside the hull). Hence all the angles are equal.
Any axis of symmetry for S must also be an axis of symmetry for the $A_i$, and hence must pass through the center C of the regular k-gon. Suppose X is a point of S in the interior of k-gon. Then it must lie inside or on some triangle $A_iA_{i+1}C$. C must be the circumcenter of $A_iA_{i+1}X$ (since it lies on the three perpendicular bisectors, which must all be axes of symmetry of S), so X must lie on the circle center C, through $A_i$ and $A_{i+1}$. But all points of the triangle $A_iA_{i+1}X$ lie strictly inside this circle, except $A_i$and $A_{i+1}$, so X cannot be in the interior of the k-gon.

**Problem 2**
**Solution** Answer: $C = 1/8$. Equality iff two $x_i$ are equal and the rest zero.

By a member of the Chinese team at the IMO - does anyone know who?
$(\Sigma x_i)^4 = (\Sigma x_i^2 + 2\Sigma_{i<j}x_ix_j)^2 >= 4 (\Sigma x_i^2) (2 \Sigma_{i<j} x_ix_j) = 8 \Sigma_{i<j} ( x_ix_j \Sigma x_k^2) >= 8 \Sigma_{i<j} x_ix_j(x_i^2 + x_j^2)$.
The second inequality is an equality only if $n - 2$ of the $x_i$ are zero. So assume that $x_3 = x_4 = ... = x_n = 0$. Then for the first inequality to be an equality we require that $(x_1^2 + x_2^2) = 2 x_1x_2$ and hence that $x_1 = x_2$. However, that is clearly also sufficient for equality.

Alternative solution:





Setting $x_1 = x_2 = 1$, $x_i = 0$ for $i > 2$ gives equality with $C = 1/8$, so, C cannot be smaller than 1/8.
We now use induction on n. For $n = 2$, the inequality with $C = 1/8$ is equivalent to $(x_1 - x_2)^4 >= 0$, which is true, with equality iff $x_1 = x_2$. So the result is true for $n = 2$.
For $n > 2$, we may take $x_1 + ... + x_n = 1$, and $x_1 <= x_2 <= ... <= x_n$. Now replace $x_1$ and $x_2$ by 0 and $x_1 + x_2$. The sum on the rhs is unchanged and the sum on the lhs is increased by $(x_1 + x_2)^3 S - (x_1^3 + x_2^3) S - x_1x_2(x_1^2 + x_2^2)$, where $S = x_3 + x_4 + ... + x_n$. But S is at least 1/3 (the critical case is $n = 3$, $x_i = 1/3$), so this is at least $x_1x_2(x_1 + x_2 - x_1^2 - x_2^2)$. This is strictly greater than 0 unless $x_1 = 0$ (when it equals 0), so the result follows by induction.

Comment. The first solution is elegant and shows clearly why the inequality is true. The second solution is more plodding, but uses an approach which is more general and can be applied in many other cases. At least with hindsight, the first solution is not as impossible to find as one might think. A little playing around soon uncovers the fact that one can get $C = 1/8$ with two $x_i$ equal and the rest zero, and that this looks like the best possible. One just has to make the jump to replacing $(x_i^2 + x_j^2)$ by $\Sigma x_k^2$. The solution is then fairly clear. Of course, that does not detract from the contestant's achievement, because I and almost everyone else who has looked at the problem failed to make that jump.

**Problem 3**
**Solution** Answer: $n/2 (n/2 + 1) = n(n + 2)/4$.
Let $n = 2m$. Color alternate squares black and white (like a chess board). It is sufficient to show that $m(m+1)/2$ white squares are necessary and sufficient to deal with all the black squares.
This is almost obvious if we look at the diagonals.
Look first at the odd-length white diagonals. In every other such diagonal, mark alternate squares (starting from the border each time, so that r+1 squares are marked in a diagonal length 2r+1). Now each black diagonal is adjacent to a picked white diagonal and hence each black square on it is adjacent to a marked white square. In all $1 + 3 + 5 + ... + m-1 + m + m-2 + ... + 4 + 2 = 1 + 2 + 3 + ... + m = m(m+1)/2$ white squares are marked. This proves sufficiency.
For necessity consider the alternate odd-length black diagonals. Rearranging, these have lengths 1, 3, 5, ... , 2m-1. A white square is only adjacent to squares in one of these alternate diagonals and is adjacent to at most 2 squares in it. So we need at least $1 + 2 + 3 + ... + m = m(m+1)/2$ white squares.

**Problem 4**
**Solution**
Answer: (1, p) for any prime p; (2, 2); (3, 3).
Evidently (1, p) is a solution for every prime p. Assume $n > 1$ and take q to be the smallest prime divisor of n. We show first that $q = p$.
Let x be the smallest positive integer for which $(p - 1)^x = - 1 \pmod q$, and y the smallest positive integer for which $(p - 1)^y = 1 \pmod q$. Certainly y exists and indeed $y < q$, since $(p - 1)^{q-1} = 1 \pmod q$. We know that $(p - 1)^n = -1 \pmod q$, so x exists also. Writing $n = sy + r$, with $0 <= r < y$, we conclude that $(p - 1)^r = -1 \pmod q$, and hence $x <= r < y$ (r cannot be zero, since 1 is not $-1 \pmod q$ ).
Now write $n = hx + k$ with $0 <= k < x$. Then $-1 = (p - 1)^n = (-1)^h(p - 1)^k \pmod q$. h cannot be even, because then $(p - 1)^k = -1 \pmod q$, contradicting the minimality of x. So

h is odd and hence $(p - 1)^k = 1 \pmod q$ with $0 <= k < x < y$. This contradicts the minimality of y unless k = 0, so n = hx. But x < q, so x = 1. So $(p - 1) = -1 \pmod q$. p and q are primes, so q = p, as claimed.
So p is the smallest prime divisor of n. We are also given that n <= 2p. So either p = n, or p = 2, n = 4. The latter does not work, so we have shown that n = p. Evidently n = p = 2 and n = p = 3 work. Assume now that p > 3. We show that there are no solutions of this type.
Expand $(p - 1)^p + 1$ by the binomial theorem, to get (since $(-1)^p = -1$): $1 + -1 + p^2 - 1/2\, p(p - 1)p^2 + p(p - 1)(p - 2)/6\, p^3 - \dots$
The terms of the form (bin coeff) $p^i$ with i >= 3 are obviously divisible by $p^3$, since the binomial coefficients are all integral. Hence the sum is $p^2$ + a multiple of $p^3$. So the sum is not divisible by $p^3$. But for p > 3, $p^{p-1}$ is divisible by $p^3$, so it cannot divide $(p - 1)^p + 1$, and there are no more solutions.

**Problem 5**
**Solution**
Let O, $O_1$, $O_2$ and r, $r_1$, $r_2$ be the centers and radii of C, $C_1$, $C_2$ respectively. Let EF meet the line $O_1O_2$ at W, and let $O_2W = x$. We need to prove that $x = r_2$.
Take rectangular coordinates with origin $O_2$, x-axis $O_2O_1$, and let O have coordinates (a, b). Notice that O and M do not, in general, lie on $O_1O_2$. Let AB meet the line $O_1O_2$ at V. We observe first that $O_2V = r_2^2/(2 r_1)$. [For example, let X be a point of intersection of $C_1$ and $C_2$ and let Y be the midpoint of $O_2X$. Then $O_1YO_2$ and $XVO_2$ are similar. Hence, $O_2V/O_2X = O_2Y/O_2O_1$.]
An expansion (or, to be technical, a homothecy) center M, factor $r/r_1$ takes $O_1$ to O and EF to AB. Hence EF is perpendicular to $O_1O_2$. Also the distance of $O_1$ from EF is $r_1/r$ times the distance of O from AB, so $(r_1 - x) = r_1/r\,(a - r_2^2/(2 r_1))$ (*).
We now need to find a. We can get two equations for a and b by looking at the distances of O from $O_1$ and $O_2$. We have:
$(r - r_1)^2 = (r_1 - a)^2 + b^2$, and
$(r - r_2)^2 = a^2 + b^2$.
Subtracting to eliminate b, we get $a = r_2^2/(2 r_1) + r - r\, r_2/r_1$. Substituting back in (*), we get $x = r_2$, as required.

**Problem 6**
**Solution**
Let c = f(0) and A be the image f(R). If a is in A, then it is straightforward to find f(a): putting a = f(y) and x = a, we get $f(a - a) = f(a) + a^2 + f(a) - 1$, so $f(a) = (1 + c)/2 - a^2/2$ (*).
The next step is to show that A - A = R. Note first that c cannot be zero, for if it were, then putting y = 0, we get: f(x - c) = f(c) + xc + f(x) - 1 (**) and hence f(0) = f(c) = 1. Contradiction. But (**) also shows that f(x - c) - f(x) = xc + (f(c) - 1). Here x is free to vary over R, so xc + (f(c) - 1) can take any value in R.
Thus given any x in R, we may find a, b in A such that x = a - b. Hence f(x) = f(a - b) = f(b) + ab + f(a) - 1. So, using (*): $f(x) = c - b^2/2 + ab - a^2/2 = c - x^2/2$.
In particular, this is true for x in A. Comparing with (*) we deduce that c = 1. So for all x in R we must have $f(x) = 1 - x^2/2$. Finally, it is easy to check that this satisfies the original relation and hence is the unique solution.





## 2000

**Problem 1**
**Solution** Angle EBA = angle BDM (because CD is parallel to AB) = angle ABM (because AB is tangent at B). So AB bisects EBM. Similarly, BA bisects angle EAM. Hence E is the reflection of M in AB. So EM is perpendicular to AB and hence to CD. So it suffices to show that MP = MQ.
Let the ray NM meet AB at X. XA is a tangent so $XA^2$ = XM.XN. Similarly, XB is a tangent, so $XB^2$ = XM.XN. Hence XA = XB. But AB and PQ are parallel, so MP = MQ.

**Problem 2**
**Solution** (B - 1 + 1/C) = B(1 - 1/B + 1/(BC) ) = B(1 + A - 1/B). Hence, (A - 1 + 1/B)(B - 1 + 1/C) = $B(A^2 - (1 - 1/B)^2) <= B\,A^2$. So the square of the product of all three <= $B\,A^2\,C\,B^2\,A\,C^2 = 1$.
Actually, that is not quite true. The last sentence would not follow if we had some negative left hand sides, because then we could not multiply the inequalities. But it is easy to deal separately with the case where (A - 1 + 1/B), (B - 1 + 1/C), (C - 1 + 1/A) are not all positive. If one of the three terms is negative, then the other two must be positive. For example, if A - 1 + 1/B < 0, then A < 1, so C - 1 + 1/A > 0, and B > 1, so B - 1 + 1/C > 0. But if one term is negative and two are positive, then their product is negative and hence less than 1.
Few people would manage this under exam conditions, but there are plenty of longer and easier to find solutions!

**Problem 3**
**Solution** Answer: k >= 1/(n-1).
Suppose k < 1/(n-1), so that $k_0 = 1/k - (n - 1) > 0$. Let X be the sum of the distances of the points from the rightmost point. If a move does not change the rightmost point, then it reduces X. If it moves the rightmost point a distance z to the right, then it reduces X by at least $z/k - (n-1)z = k_0 z$. X cannot be reduced below nil. So the total distance moved by the rightmost point is at most $X_0/k_0$, where $X_0$ is the initial value of X.
Conversely, suppose k >= 1/(n-1), so that $k_1 = (n - 1) - 1/k >= 0$. We always move the leftmost point. This has the effect of moving the rightmost point z > 0 and increasing X by $(n-1)z - z/k = k_1z >= 0$. So X is never decreased. But $z >= k\,X/(n-1) >= k\,X_0/(n-1) > 0$. So we can move the rightmost point arbitarily far to the right (and hence all the points, since another n - 1 moves will move the other points to the right of the rightmost point).

**Problem 4**
**Solution** Answer: 12. Place 1, 2, 3 in different boxes (6 possibilities) and then place n in the same box as its residue mod 3. Or place 1 and 100 in different boxes and 2 - 99 in the third box (6 possibilities).
Let $H_n$ be the corresponding result that for cards numbered 1 to n the only solutions are by residue mod 3, or 1 and n in separate boxes and 2 to n - 1 in the third box. It is easy to check that they *are* solutions. $H_n$ is the assertion that there are no others. $H_3$ is obviously true (although the two cases coincide). We now use induction on n. So suppose that the result is true for n and consider the case n + 1.
Suppose n + 1 is alone in its box. If 1 is not also alone, then let N be the sum of the largest cards in each of the boxes not containing n + 1. Since n + 2 <= N <= n + (n - 1) = 2n - 1, we can achieve the same sum N as from a different pair of boxes as (n + 1) +

(N - n - 1). Contradiction. So 1 must be alone and we have one of the solutions envisaged in $H_{n+1}$.
If n + 1 is not alone, then if we remove it, we must have a solution for n. But that solution cannot be the n, 1, 2 to n - 1 solution. For we can easily check that none of the three boxes will then accomodate n + 1. So it must be the mod 3 solution. We can easily check that in this case n + 1 must go in the box with matching residue, which makes the (n + 1) solution the other solution envisaged by $H_{n+1}$. That completes the induction.

My much more *plodding* solution (which I was quite pleased with until I saw the more elegant solution above) follows. It took about half-an-hour and shows the kind of kludge one is likely to come up with under time pressure in an exam!
With a suitable labeling of the boxes as A, B, C, there are 4 cases to consider:
Case 1: A contains 1; B contains 2; C contains 3
Case 2: A contains 1,2
Case 3: A contains 1, 3; B contains 2
Case 4: A contains 1; B contains 2, 3.
We show that Cases 1 and 4 each yield just one possible arrangement and Cases 2 and 3 none.
In Case 1, it is an easy induction that n must be placed in the same box as its residue (in other words numbers with residue 1 mod 3 go into A, numbers with residue 2 go into B, and numbers with residue 0 go into C). For (n + 1) + (n - 2) = n + (n - 1). Hence n + 1 must go in the same box as n - 2 (if they were in different boxes, then we would have two pairs from different pairs of boxes with the same sum). It is also clear that this is a possible arrangement. Given the sum of two numbers from different boxes, take its residue mod 3. A residue of 0 indicates that the third (unused) box was C, a residue of 1 indicates that the third box was A, and a residue of 2 indicates that the third box was B. Note that this unique arrangement gives 6 ways for the question, because there are 6 ways of arranging 1, 2 and 3 in the given boxes.
In Case 2, let n be the smallest number not in box A. Suppose it is in box B. Let m be the smallest number in the third box, C. m - 1 cannot be in C. If it is in A, then m + (n - 1) = (m - 1) + n. Contradiction (m is in C, n - 1 is in A, so that pair identifies B as the third box, but m - 1 is in A and n is in B, identifying C). So m - 1 must be in B. But (m - 1) + 2 = m + 1. Contradiction. So Case 2 is not possible.
In Case 3, let n be the smallest number in box C, so n - 1 must be in A or B. If n - 1 is in A, then (n - 1) + 2 = n + 2. Contradiction (a sum of numbers in A and B equals a sum from C and A). If n - 1 is in B, then (n - 1) + 3 = n + 2. Contradiction ( a sum from B and A equals a sum from C and B). So Case 3 is not possible.
In Case 4, let n be the smallest number in box C. n - 1 cannot be in A, or (n - 1) + 2 = 3 + n (pair from A, B with same sum as pair from B, C), so n - 1 must be in B. Now n + 1 cannot be in A (or (n + 1) + 2 = 3 + n), or in B or C (or 1 + (n + 1) = 2 + n). So n + 1 cannot exist and hence n = 100. It is now an easy induction that all of 4, 5, ... 98 must be in B. For given that m is in B, if m + 1 were in A, we would have 100 + m = 99 + (m + 1). But this arrangement (1 in A, 2 - 99 in B, 100 in C) is certainly possible: sums 3 - 100 identify C as the third box, sum 101 identifies B as the third box, and sums 102-199 identify A as the third box. Finally, as in Case 1, this unique arrangement corresponds to 6 ways of arranging the cards in the given boxes.

**Problem 5**
**Solution** Answer: yes.





Note that for b odd we have $2^{ab} + 1 = (2^a + 1)(2^{a(b-1)} - 2^{a(b-2)} + ... + 1)$, and so $2^a + 1$ is a factor of $2^{ab} + 1$. It is sufficient therefore to find m such that (1) m has only a few distinct prime factors, (2) $2^m + 1$ has a large number of distinct prime factors, (3) m divides $2^m + 1$. For then we can take k, a product of enough distinct primes dividing $2^m + 1$ (but not m), so that km has exactly 2000 factors. Then km still divides $2^m + 1$ and hence $2^{km} + 1$.
The simplest case is where m has only one distinct prime factor p, in other words it is a power of p. But if p is a prime, then p divides $2^p - 2$, so the only p for which p divides $2^p + 1$ is 3. So the questions are whether $a_h = 2^m + 1$ is (1) divisible by $m = 3^h$ and (2) has a large number of distinct prime factors.
$a_{h+1} = a_h(2^{2m} - 2^m + 1)$, where $m = 3^h$. But $2^m = (a_h - 1)$, so $a_{h+1} = a_h(a_h^2 - 3 a_h + 3)$. Now $a_1 = 9$, so an easy induction shows that $3^{h+1}$ divides $a_h$, which answers (1) affirmatively. Also, since $a_h$ is a factor of $a_{h+1}$, any prime dividing $a_h$ also divides $a_{h+1}$. Put $a_h = 3^{h+1}b_h$. Then $b_{h+1} = b_h(3^{2h+1}b_h^2 - 3^{h+2}b_h + 1)$. Now $(3^{2h+1}b_h^2 - 3^{h+2}b_h + 1) > 1$, so it must have some prime factor p > 1. But p cannot be 3 or divide $b_h$ (since $(3^{2h+1}b_h^2 - 3^{h+2}b_h + 1)$ is a multiple of $3b_h$ plus 1), so $b_{h+1}$ has at least one prime factor p > 3 which does not divide $b_h$. So $b_{h+1}$ has at least h distinct prime factors greater than 3, which answers (2) affirmatively. But that is all we need. We can take m in the first paragraph above to be $3^{2000}$: (1) m has only one distinct prime factor, (2) $2^m + 1 = 3^{2001} b_{2000}$ has at least 1999 distinct prime factors other than 3, (3) m divides $2^m + 1$. Take k to be a product of 1999 distinct prime factors dividing $b_{2000}$. Then N = km is the required number with exactly 2000 distinct prime factors which divides $2^N + 1$.

**Problem 6**
**Solution** Let O be the centre of the incircle. Let the line parallel to $A_1A_2$ through $L_2$ meet the line $A_2O$ at X. We will show that X is the reflection of $K_2$ in $L_2L_3$. Let $A_1A_3$ meet the line $A_2O$ at $B_2$. Now $A_2K_2$ is perpendicular to $K_2B_2$ and $OL_2$ is perpendicular to $L_2B_2$, so $A_2K_2B_2$ and $OL_2B_2$ are similar. Hence $K_2L_2/L_2B_2 = A_2O/OB_2$. But $OA_3$ is the angle bisector in the triangle $A_2A_3B_2$, so $A_2O/OB_2 = A_2A_3/B_2A_3$.
Take $B'_2$ on the line $A_2O$ such that $L_2B_2 = L_2B'_2$ ($B'_2$ is distinct from $B_2$ unless $L_2B_2$ is perpendicular to the line). Then angle $L_2B'_2X$ = angle $A_3B_2A_2$. Also, since $L_2X$ is parallel to $A_2A_1$, angle $L_2XB'_2$ = angle $A_3A_2B_2$. So the triangles $L_2XB'_2$ and $A_3A_2B_2$ are similar. Hence $A_2A_3/B_2A_3 = XL_2/B_2'L_2 = XL_2/B_2L_2$ (since $B'_2L_2 = B_2L_2$).
Thus we have shown that $K_2L_2/L_2B_2 = XL_2/B_2L_2$ and hence that $K_2L_2 = XL_2$. $L_2X$ is parallel to $A_2A_1$ so angle $A_2A_1A_3$ = angle $A_1L_2X$ = angle $L_2XK_2$ + angle $L_2K_2X$ = 2 angle $L_2XK_2$ (isosceles). So angle $L_2XK_2$ = 1/2 angle $A_2A_1A_3$ = angle $A_2A_1O$. $L_2X$ and $A_2A_1$ are parallel, so $K_2X$ and $OA_1$ are parallel. But $OA_1$ is perpendicular to $L_2L_3$, so $K_2X$ is also perpendicular to $L_2L_3$ and hence X is the reflection of $K_2$ in $L_2L_3$.
Now the angle $K_3K_2A_1$ = angle $A_1A_2A_3$, because it is 90 - angle $K_3K_2A_2$ = 90 - angle $K_3A_3A_2$ ($A_2A_3K_2K_3$ is cyclic with $A_2A_3$ a diameter) = angle $A_1A_2A_3$. So the reflection of $K_2K_3$ in $L_2L_3$ is a line through X making an angle $A_1A_2A_3$ with $L_2X$, in other words, it is the line through X parallel to $A_2A_3$.
Let $M_i$ be the reflection of $L_i$ in $A_iO$. The angle $M_2XL_2$ = 2 angle $OXL_2$ = 2 angle $A_1A_2O$ (since $A_1A_2$ is parallel to $L_2X$) = angle $A_1A_2A_3$, which is the angle betwee $L_2X$ and $A_2A_3$. So $M_2X$ is parallel to $A_2A_3$, in other words, $M_2$ lies on the reflection of $K_2K_3$ in $L_2L_3$.
If follows similarly that $M_3$ lies on the reflection. Similarly, the line $M_1M_3$ is the reflection of $K_1K_3$ in $L_1L_3$, and the line $M_1M_2$ is the reflection of $K_1K_2$ in $L_1L_2$ and hence the triangle formed by the intersections of the three reflections is just $M_1M_2M_3$.

## 2001

**Problem 1**

**Solution** Take D on the circumcircle with AD parallel to BC. Angle CBD = angle BCA, so angle ABD >= 30°. Hence angle AOD >= 60°. Let Z be the midpoint of AD and Y the midpoint of BC. Then AZ >= R/2, where R is the radius of the circumcircle. But AZ = YX (since AZYX is a rectangle).
Now O cannot coincide with Y (otherwise angle A would be 90° and the triangle would not be acute-angled). So OX > YX >= R/2. But XC = YC - YX < R - YX <= R/2. So OX > XC.
Hence angle COX < angle OCX. Let CE be a diameter of the circle, so that angle OCX = angle ECB. But angle ECB = angle EAB and angle EAB + angle BAC = angle EAC = 90°, since EC is a diameter. Hence angle COX + angle BAC < 90°.

**Problem 2**

**Solution** A not particularly elegant, but fairly easy, solution is to use Cauchy: $(\Sigma xy)^2 <= \Sigma x^2 \Sigma y^2$.
To get the inequality the right way around we need to take $x^2 = a/a'$ [to be precise, we are taking $x_1^2 = a/a'$, $x_2^2 = b/b'$, $x_3^2 = c/c'$.]. Take $y^2 = a a'$, so that $xy = a$. Then we get $\Sigma a/a' >= (\Sigma a)^2/\Sigma a a'$.
Evidently we need to apply Cauchy again to deal with $\Sigma a a'$. This time we want $\Sigma a a' <=$ something. The obvious X=a, Y=a' does not work, but if we put $X=a^{1/2}$, $Y=a^{1/2}a'$, then we have $\Sigma a a' <= (\Sigma a)^{1/2} (\Sigma a a'^2)^{1/2}$. So we get the required inequality provided that $(\Sigma a)^{3/2} >= (\Sigma a a'^2)^{1/2}$ or $(\Sigma a)^3 >= \Sigma a a'^2$.
Multiplying out, this is equivalent to: $3(ab^2 + ac^2 + ba^2 + bc^2 + ca^2 + cb^2) >= 18abc$, or $a(b - c)^2 + b(c - a)^2 + c(a - b)^2 >= 0$, which is clearly true.

**Problem 3**

**Solution** Notice first that the result is not true for a 20 x 20 array. Make 20 rectangles each 2 x 10, labelled 1, 2, ... , 20. Divide the 20 x 20 array into four quadrants (each 10 x 10). In each of the top left and bottom right quadrants, place 5 rectangles horizontally. In each of the other two quadrants, place 5 rectangles vertically. Now each row intersects 5 vertical rectangles and 1 horizontal. In other words, it contains just 6 different numbers. Similarly each column. But any given number is in either 10 rows and 2 columns or vice versa, so no number is in 3 rows and 3 columns. [None of this is necessary for the solution, but it helps to show what is going on.]
Returning to the 21 x 21 array, assume that an arrangement is possible with no integer in at least 3 rows and at least 3 columns. Color a cell white if its integer appears in 3 or more rows and black if its integer appears in only 1 or 2 rows. We count the white and black squares.
Each row has 21 cells and at most 6 different integers. 6 x 2 < 21, so every row includes an integer which appears 3 or more times and hence in at most 2 rows. Thus at most 5 different integers in the row appear in 3 or more rows. Each such integer can appear at most 2 times in the row, so there are at most 5 x 2 = 10 white cells in the row. This is true for every row, so there are at most 210 white cells in total.
Similarly, any given column has at most 6 different integers and hence at least one appears 3 or more times. So at most 5 different integers appear in 2 rows or less. Each such integer can occupy at most 2 cells in the column, so there are at most 5 x 2 = 10 black cells in the column. This is true for every column, so there are at most 210 black cells in total.





This gives a contradiction since 210 + 210 < 441.
*Comment. This looks easy, but (like question 6) I found it curiously difficult (it took me well over 2 hours). For a while I could not see how to do better than a 12 x 12 array (with 2 rows of 1s, then 2 rows of 2s etc), which was disorienting. Then I got the argument almost right, but not quite right, which took more time.*
*The original question was phrased in terms of 21 boys and 21 girls in a competition with an unknown number of problems. Each boy, girl pair solved at least one problem. Each competitor solved at most 6 problems. One had to show that some problem was solved by at least 3 boys and at least 3 girls. The recasting in the terms above is almost immediate.*
*Equally, one can easily recast the solution above into the competition format. Take any boy $B_0$. At least one of the questions he attempts must be attempted by 3 or more girls (because he attempts at most 6 questions and there are more than 6x2 girls). Hence he attempts at most 5 questions which are only attempted by less than 3 girls. So at most 5 x 2 = 10 of the 21 pairs ($B_0$, G) attempt a question attempted by less than 3 girls. So at most 210 of the 441 pairs pairs (B, G) attempt such a question. Similarly, at most 210 pairs (B, G) attempt a question attempted by less than 3 boys. Hence at least 21 pairs (B, G) attempt a question attempted by 3 or more girls and 3 or more boys. So there must be at least one such question.*
*Note that the arguments above generalise immediately to show that in a 4N+1 by 4N+1 array with at most N+1 different integers in each row and column, there is some integer that appears in at least 3 rows and 3 columns, but this is not true for a 4N by 4N array.*

**Problem 4**

**Solution** This is a simple application of the pigeon hole principle.
The sum of all m! distinct residues mod m! is not divisible by m! because m! is even (since m > 1). [The residues come in pairs a and m! - a, except for m!/2.].
However, the sum of all f(x) as x ranges over all m! permutations is $1/2\ (m+1)!\ \Sigma n_i$, which is divisible by m! (since m+1 is even). So at least one residue must occur more than once among the f(x).

**Problem 5**

**Solution** Answer: 80°.
This is an inelegant solution, but I did get it fast! Without loss of generality we can take length AB = 1. Take angle ABY = x. Note that we can now solve the two triangles AXB and AYB. In particular, using the sine rule, BX = sin 30°/sin(150°-2x), AY = sin x/sin(120°-x), YB = sin 60°/sin(120°-x). So we have an equation for x.
Using the usual formula for sin(a + b) etc, and writing s = sin x, c = cos x, we get: $2\sqrt{3}\, s^2c - 4sc - 2\sqrt{3}\, c^3 + 2\sqrt{3}\, c^2 + 6sc - 2s - \sqrt{3} = 0$ or $-\sqrt{3}\,(4c^3 - 2c^2 - 2c + 1) = 2s(2c^2 -3c + 1)$. This has a common factor 2c - 1. So c = 1/2 or $-\sqrt{3}\,(2c^2 - 1) = 2s(c - 1)$ (*).
c = 1/2 means x = 60° or angle B = 120°. But in that case the sides opposite A and B are parallel and the triangle is degenerate (a case we assume is disallowed). So squaring (*) and using $s^2 = 1 - c^2$, we get: $16c^4 - 8c^3 - 12c^2 + 8c - 1 = 0$. This has another factor 2c - 1. Dividing that out we get: $8c^3 - 6c + 1 = 0$. But we remember that $4c^3 - 3c = \cos 3x$, so we conclude that cos 3x = -1/2. That gives x = 40°, 80°, 160°, 200°, 280°, 320°. But we require that x < 60° to avoid degeneracy. Hence the angle B = 2x = 80°.
*I subsequently found this geometric solution on the official Wolfram site (Wolfram was one of the sponsors of IMO 2001). I cannot say it is much easier, but at least it is geometric.*

Extend AB to X' with BX' = BX. Extend AY to Z with YZ = YB. Then AZ = AY + YZ = AY + YB = AB + BX = AB + BX' = AX'. Angle A = $60^o$, so AZX' is equilateral.
Use B also to denote the angle at B. Then angle YBX = B/2. Also angle BXX' + angle BX'X = B. The triangle is isosceles by construction, so angle BX'X = B/2. Hence angle XX'Z = $60^o$ - B/2. X lies on the bisector of A and AZ = AX', so XZ = XX'. Hence XZX' = $60^o$ - B/2 also. But angle Z = $60^o$, so angle YZX = B/2 = angle YBX.
Now YZ = YB, so angle YZB = angle YBZ. Hence angle XZB = angle XBX (they are the difference of pairs of equal angles). If X does not lie on BZ, then we can conclude that XB = XZ.
In that case, since XZ = XX', we have XB = XX'. But already XB = BX' (by construction), so BXX' is equilateral and hence B/2 = $60^o$. But then angle B + angle A = $180^o$, so the triangle ABC is degenerate (with C at infinity), which we assume is disallowed. Hence X must lie on BZ, which means Z = C and angle B = 2 angle C. Hence angle B = 80, angle C = 40.

**Problem 6**
**Solution** Note first that KL+MN > KM+LN > KN+LM, because (KL+MN) - (KM+LN) = (K - N)(L - M) > 0 and (KM+LN) - (KN-LM) = (K - L)(M - N) > 0.
Multiplying out and rearranging, the relation in the question gives $K^2 - KM + M^2 = L^2 + LN + N^2$. Hence $(KM + LN)(L^2 + LN + N^2) = KM(L^2 + LN + N^2) + LN(K^2 - KM + M^2) = KML^2 + KMN^2 + K^2LN + LM^2N = (KL + MN)(KN + LM)$. In other words (KM + LN) divides (KL + MN)(KN + LM).
Now suppose KL + MN is prime. Since it greater than KM + LN, it can have no common factors with KM + LN. Hence KM + LN must divide the smaller integer KN + LM. Contradiction.
*Comment. This looks easy, but in fact I found it curiously difficult. It is easy to go around in circles getting nowhere. Either I am getting older, or this is harder than it looks!*
*Note that it is not hard to find K, L, M, N satisfying the condition in the question. For example 11, 9, 5, 1.*

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## 2002

**Problem 1**

**Solution.** Let $a_i$ be the number of blue members (h, k) in S with h = i, and let $b_i$ be the number of blue members (h, k) with k = i. It is sufficient to show that $b_0, b_1, \dots, b_{n-1}$ is a rearrangement of $a_0, a_1, \dots, a_{n-1}$ (because the number of type 1 subsets is the product of the $a_i$ and the number of type 2 subsets is the product of the $b_i$).

Let $c_i$ be the largest k such that (i, k) is red. If (i, k) is blue for all k then we put $c_i = -1$. Note that if i < j, then $c_i >= c_j$, since if (j, $c_i$ ) is red, then so is (i, $c_i$ ). Note also that (i, k) is red for k <= $c_i$, so the sequence $c_0, c_1, \dots, c_{n-1}$ completely defines the coloring of S.

Let $S_i$ be the set with the sequence $c_0, c_1, \dots, c_i, -1, \dots, -1$, so that $S_{n-1}$ = S. We also take $S_{-1}$ as the set with the sequence -1, -1, ... , -1, so that all its members are blue. We show that the rearrangement result is true for $S_{-1}$ and that if it is true for $S_i$ then it is true for $S_{i+1}$. It is obvious for $S_{-1}$, because both $a_i$ and $b_i$ are n, n-1, ... , 2, 1. So suppose it is true for $S_i$ (where i < n-1). The only difference between the $a_j$ for $S_i$ and for $S_{i+1}$ is that $a_{i+1}$ =





n-i-1 for $S_i$ and $(n-i-1)-(c_{i+1}+1)$ for $S_{i+1}$. In other words, the number n-i-1 is replaced by the number n-i-c-2, where c = $c_{i+1}$. The difference in the $b_j$ is that 1 is deducted from each of $b_0, b_1, \dots, b_c$. But these numbers are just n-i-1, n-i-1, n-i-2, ... , n-i-c-1. So the effect of deducting 1 from each is to replace n-i-1 by n-i-c-2, which is the same change as was made to the $a_j$. So the rearrangement result also holds for $S_{i+1}$. Hence it holds for S.

**Problem 2**

**Solution.** D is the midpoint of the arc AB, so angle DOB = 1/2 angle AOB = angle ACB. Hence DO is parallel to AC. But OJ is parallel to AC, so AJOD is a parallelogram. Hence AJ = OD. Since EF is the perpendicular bisector of AO, we have AE = AF = OE, so A is the center of the circle through E, F and J. Hence angle EFJ = 1/2 angle EAJ. But angle EAJ = angle EAC (same angle) = angle EFC. Hence J lies on the bisector of angle EFC.

Since EF is perpendicular to AO, A is the midpoint of the arc EF. Hence angle ACE = angle ACF, so J lies on the bisector of angle ECF. Hence J is the incenter.

**Problem 3**

**Solution.** Answer: m = 5, n = 3.

Obviously m > n. Take polynomials q(x), r(x) with integer coefficients and with degree r(x) < n such that $x^m + x - 1 = q(x)(x^n + x^2 - 1) + r(x)$. Then $x^n + x^2 - 1$ divides r(x) for infinitely many positive integers x. But for sufficiently large x, $x^n + x^2 - 1 > r(x)$ since r(x) has smaller degree. So r(x) must be zero. So $x^m + x - 1$ factorises as $q(x)(x^n + x^2 - 1)$, where $q(x) = x^{m-n} + a_{m-n-1}x^{m-n-1} + \dots + a_0$.

We have $(x^m + x - 1) = x^{m-n}(x^n + x^2 - 1) + (1 - x)(x^{m-n+1} + x^{m-n} - 1)$, so $(x^n + x^2 - 1)$ must divide $(x^{m-n+1} + x^{m-n} - 1)$. So, in particular, m >= 2n-1. Also $(x^n + x^2 - 1)$ must divide $(x^{m-n+1} + x^{m-n} - 1) - x^{m-2n+1}(x^n + x^2 - 1) = x^{m-n} - x^{m-2n+3} + x^{m-2n+1} - 1$ (*). If m = 2n-1, this is $x^{n-1} - x^2$. If n = 3, this is 0 and indeed we find m = 5, n = 3 gives an identity. If n > 3, then it is $x^2(x^{n-3} - 1)$. But this has no roots in the interval (0, 1), whereas $x^n + x^2 - 1$ has at least one (because it is -1 at x = 0 and +1 at x = 1), so $x^n + x^2 - 1$ cannot be a factor.

If m > 2n-1, then (*) has four terms and factorises as $(x - 1)(x^{m-n-1} + x^{m-n-2} + \dots + x^{m-2n+3} + x^{m-2n} + x^{m-2n-1} + \dots + 1)$. Again, this has no roots in the interval (0, 1), whereas $x^n + x^2 - 1$ has at least one, so $x^n + x^2 - 1$ cannot be a factor.

**Problem 4**

**Solution.** $d_{k+1-m}$ <= n/m. So d < $n^2$(1/(1.2) + 1/(2.3) + 1/(3.4) + ... ). The inequality is certainly strict because d has only finitely many terms. But 1/(1.2) + 1/(2.3) + 1/(3.4) + ... = (1/1 - 1/2) + (1/2 - 1/3) + (1/3 - 1/4) + ... = 1. So d < $n^2$.

Obviously d divides $n^2$ for n prime. Suppose n is composite. Let p be the smallest prime dividing n. Then $d > n^2/p$. But the smallest divisor of $n^2$ apart from 1 is p, so if d divides $n^2$, then $d <= n^2/p$. So d cannot divide $n^2$ for n composite.

**Problem 5**

**Solution.** Answer: there are three possible functions: (1) $f(x) = 0$ for all x; (2) $f(x) = 1/2$ for all x; or (3) $f(x) = x^2$.

Put $x = y = 0$, $u = v$, then $4 f(0) f(u) = 2 f(0)$. So either $f(u) = 1/2$ for all u, or $f(0) = 0$. $f(u) = 1/2$ for all u is certainly a solution. So assume $f(0) = 0$.

Putting $y = v = 0$, $f(x) f(u) = f(xu)$ (*). In particular, taking $x = u = 1$, $f(1)^2 = f(1)$. So $f(1) = 0$ or 1. Suppose $f(1) = 0$. Putting $x = y = 1$, $v = 0$, we get $0 = 2f(u)$, so $f(x) = 0$ or all x. That is certainly a solution. So assume $f(1) = 1$.

Putting $x = 0$, $u = v = 1$ we get $2 f(y) = f(y) + f(-y)$, so $f(-y) = f(y)$. So we need only consider f(x) for x positive. We show next that $f(r) = r^2$ for r rational. The first step is to show that $f(n) = n^2$ for n an integer. We use induction on n. It is true for n = 0 and 1. Suppose it is true for n-1 and n. Then putting $x = n$, $y = u = v = 1$, we get $2f(n) + 2 = f(n-1) + f(n+1)$, so $f(n+1) = 2n^2 + 2 - (n-1)^2 = (n+1)^2$ and it is true for n+1. Now (*) implies that $f(n) f(m/n) = f(m)$, so $f(m/n) = m^2/n^2$ for integers m, n. So we have established $f(r) = r^2$ for all rational r.

From (*) above, we have $f(x^2) = f(x)^2 >= 0$, so f(x) is always non-negative for positive x and hence for all x. Putting $u = y$, $v = x$, we get $( f(x) + f(y) )^2 = f(x^2 + y^2)$, so $f(x^2 + y^2) = f(x)^2 + 2f(x)f(y) + f(y)^2 >= f(x)^2 = f(x^2)$. For any $u > v > 0$, we may put $u = x^2 + y^2$, $v = x^2$ and hence $f(u) >= f(v)$. In other words, f is an increasing function.

So for any x we may take a sequence of rationals $r_n$ all less than x we converge to x and another sequence of rationals $s_n$ all greater than x which converge to x. Then $r_n^2 = f(r_n) <= f(x) <= f(s_n) = s_n^2$ for all x and hence $f(x) = x^2$.

**Problem 6**

**Solution.** Denote the circle center $O_i$ by $C_i$. The tangents from $O_1$ to $C_i$ contain an angle 2x where $\sin x = 1/O_1O_i$. So $2x > 2/O_1O_i$. These double sectors cannot overlap, so $\Sigma\ 2/O_1O_i < \pi$. Adding the equations derived from $O_2, O_3, \dots$ we get $4\Sigma\ O_iO_j < n\pi$, so $\Sigma\ O_iO_j < n\pi/4$, which is not quite good enough.

There are two key observations. The first is that it is better to consider the angle $O_iO_1O_j$ than the angle between the tangents to a single circle. It is not hard to show that this angle must exceed both $2/O_1O_i$ and $2/O_1O_j$. For consider the two common tangents to $C_1$ and $C_i$ which intersect at the midpoint of $O_1O_i$. The angle between the center line and one of the tangents is at least $2/O_1O_i$. No part of the circle $C_j$ can cross this line, so its center $O_j$ cannot cross the line parallel to the tangent through $O_1$. In other word, angle $O_iO_1O_j$ is at least $2/O_1O_i$. A similar argument establishes it is at least $2/O_1O_j$.





Now consider the convex hull of the n points $O_i$. m <= n of these points form the convex hull and the angles in the convex m-gon sum to $(2m-2)\pi$. That is the second key observation. That gains us not one but two amounts $\pi/4$. However, we lose one back. Suppose $O_1$ is a vertex of the convex hull and that its angle is $\theta_1$. Suppose for convenience that the rays $O_1O_2, O_1O_3, \dots , O_1O_n$ occur in that order with $O_2$ and $O_n$ adjacent vertices to $O_1$ in the convex hull. We have that the n-2 angles between adjacent rays sum to $\theta_1$. So we have $\Sigma\ 2/O_1O_i < \theta_1$, where the sum is taken over only n-2 of the i, not all n-1. But we can choose which i to drop, because of our freedom to choose either distance for each angle. So we drop the longest distance $O_1O_i$. [If $O_1O_k$ is the longest, then we work outwards from that ray. Angle $O_{k-1}O_1O_k > 2/O_1O_{k-1}$, and angle $O_kO_1O_{k+1} > 2/O_1O_{k+1}$ and so on.]

We now sum over all the vertices in the convex hull. For any centers $O_i$ inside the hull we use the $\Sigma_j\ 2/O_iO_j < \pi$ which we established in the first paragraph, where the sum has all n-1 terms. Thus we get $\Sigma_{i,j}\ 2/O_iO_j < (n-2)\pi$, where for vertices i for which $O_i$ is a vertex of the convex hull the sum is only over n-2 values of j and excludes $2/O_iO_{\max i}$ where $O_{\max i}$ denotes the furthest center from $O_i$.

Now for $O_i$ a vertex of the convex hull we have that the sum over all j, $\Sigma 2/O_iO_j$, is the sum $\Sigma'$ over all but j = max i plus at most $1/(n-2)\ \Sigma'$. In other words we must increase the sum by at most a factor $(n-1)/(n-2)$ to include the missing term. For $O_i$ not a vertex of the hull, obviously no increase is needed. Thus the full sum $\Sigma_{i,j}\ 2/O_iO_j < (n-1)\pi$. Hence $\Sigma_{i<j}\ 1/O_iO_j < (n-1)\pi/4$ as required.

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## 2003

**Problem 1**
**Solution** Having found $x_1, x_2, \dots , x_k$ there are $k\cdot101\cdot100$ forbidden values for $x_{k+1}$ of the form $x_i + a_m - a_n$ with m and n unequal and another k forbidden values with m = n. Since $99\cdot101\cdot100 + 99 = 10^6 - 1$, we can successively choose 100 distinct $x_i$.

**Problem 2**
**Solution Answer** $(m, n) = (2k, 1), (k, 2k)$ or $(8k^4 - k, 2k)$. The denominator is $2mn^2 - n^3 + 1 = n^2(2m - n) + 1$, so $2m >= n > 0$. If n = 1, then m must be even, in other words, we have the solution (m, n) = (2k, 1).

So assume n > 1. Put $h = m^2/(2mn^2 - n^3 + 1)$. Then we have a quadratic equation for m, namely $m^2 - 2hn^2m + (n^3 - 1)h = 0$. This has solutions $hn^2 +- N$, where N is the positive square root of $h^2n^4 - hn^3 + h$. Since $n > 1$, $h \geq 1$, N is certainly real. But the sum and product of the roots are both positive, so both roots must be positive. The sum is an integer, so if one root is a positive integer, then so is the other.

The larger root $hn^2 + N$ is greater than $hn^2$, so the smaller root $< h(n^3 - 1)/(hn^2) < n$. But note that if $2m - n > 0$, then since $h > 0$, we must have the denominator $(2m - n)n^2 + 1$ smaller than the numerator and hence m > n. So for the smaller root we cannot have 2m - n > 0. But 2m - n must be non-negative (since h is positive), so 2m - n = 0 for the smaller root. Hence $hn^2 - N = n/2$. Now $N^2 = (hn^2 - n/2)^2 = h^2n^4 - hn^3 + h$, so $h = n^2/4$.

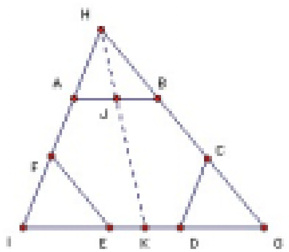
Thus n must be even. Put n = 2k and we get the solutions (m, n) = (k, 2k) and ($8k^4$ - k, 2k).

We have shown that any solution must be of one of the three forms given, but it is trivial to check that they are all indeed solutions.

**Problem 3**

**Solution** We use bold to denote vectors, so **AB** means the vector from A to B. We take some arbitrary origin and write the vector **OA** as **A** for short. Note that the vector to the midpoint of AB is (**A** + **B**)/2, so the vector from the midpoint of DE to the midpoint of AB is (**A** + **B** - **D** - **E**)/2. So the starting point is $|\mathbf{A} + \mathbf{B} - \mathbf{D} - \mathbf{E}| \geq \sqrt{3}\ ( |\mathbf{A} - \mathbf{B}| + |\mathbf{D} - \mathbf{E}| )$ and two similar equations. The key is to notice that by the triangle inequality we have $|\mathbf{A} - \mathbf{B}| + |\mathbf{D} - \mathbf{E}| \geq |\mathbf{A} - \mathbf{B} - \mathbf{D} + \mathbf{E}|$ with equality iff the opposite sides AB and DE are parallel. Thus we get $|\mathbf{DA} + \mathbf{EB}| \geq \sqrt{3}\ |\mathbf{DA} - \mathbf{EB}|$. Note that DA and EB are diagonals. Squaring, we get $\mathbf{DA}^2 + 2\ \mathbf{DA.EB} + \mathbf{EB}^2 \geq 3(\mathbf{DA}^2 - 2\ \mathbf{DA.EB} + \mathbf{EB}^2)$, or $\mathbf{DA}^2 + \mathbf{EB}^2 \leq 4\ \mathbf{DA.EB}$. Similarly, we get $\mathbf{EB}^2 + \mathbf{FC}^2 \leq 4\ \mathbf{EB.FC}$ and $\mathbf{FC}^2 + \mathbf{AD}^2 \leq 4\ \mathbf{FC.AD} = -4\ \mathbf{FC.DA}$. Adding the three equations gives $2(\mathbf{DA} - \mathbf{EB} + \mathbf{FC})^2 \leq 0$. So it must be zero, and hence **DA** - **EB** + **FC** = 0 and opposite sides of the hexagon are parallel.

Note that **DA** - **EB** + **FC** = **A** - **D** - **B** + **E** + **C** - **F** = **BA** + **DC** + **FE**. So **BA** + **DC** + **FE** = 0. In other words, the three vectors can form a triangle.




Since EF is parallel to BC, if we translate EF along the vector **ED** we get CG, an extension of BC. Similarly, if we translate AB along the vector **BC** we get an extension of ED. Since **BA**, **DC** and **FE** form a triangle, AB must translate to DG. Thus HAB and CDG are congruent. Similarly, if we take AF and DE to intersect at I, the triangle FIE is also congruent (and similarly oriented) to HAB and CDG. Take J, K as the midpoints of AB, ED. HIG and HAB are equiangular and hence similar. IE = DG and K is the midpoint of ED, so K is also the midpoint of IG. Hence HJ is parallel to HK, so H, J, K are collinear.

Hence HJ/AB = HK/IG = (HK - HJ)/(IG - AB) = JK/(AB + ED) = ½ √3. Similarly, each of the medians of the triangle HAB is ½ √3 times the corresponding side. We will show that this implies it is equilateral. The required result then follows immediately.

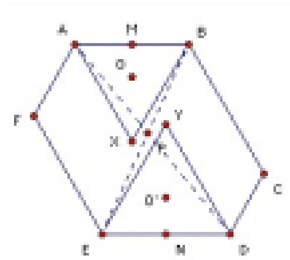
Suppose a triangle has side lengths a, b, c and the length of the median to the midpoint of side length c is m. Then applying the cosine rule twice we get $m^2 = a^2/2 + b^2/2 - c^2/4$.





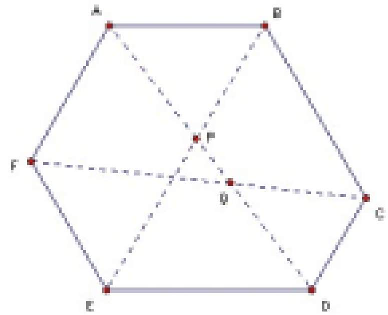
So if $m^2 = ¾\ c^2$, it follows that $a^2 + b^2 = 2c^2$. Similarly, $b^2 + c^2 = 2a^2$. Subtracting, a = c. Similarly for the other pairs of sides.

*An alternative (and rather more elegant) solution sent my some anonymous contestants at the IMO is as follows*



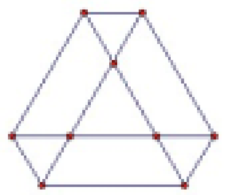

Let the diagonals AD and BE meet at P. We show that angle APB <= 60°. Suppose angle APB > 60°. Take X and Y inside the hexagon so that ABX and DEY are equilateral (as shown). Then since angle APB > angle AXB, P lies inside the circumcircle of ABX (which we take to have center O, radius r). Similarly, it lies inside the circumcircle of DEY (which we take to have center O' radius r'), so these circles must meet and hence OO' < r + r'. Now √3 (AB + DE)/2 = MN (where M, N are the midpoints of AB, DE) ≤ MO + OO' + O'N < r/2 + (r + r') + r'/2 = (3/2)(r + r') = √3 (AB + DE)/2. Contradiction.

The same argument applies to any two long diagonals. Hence the angles must all be 60°. Also we must have MP ≤ MX with equality iff P = X, and similarly NP ≤ NY with equality iff P = Y. So MN ≤ MP + PN ≤ MX + NY = √3 (AB + DE)/2 = MN. Hence we have equality and so P = X = Y.



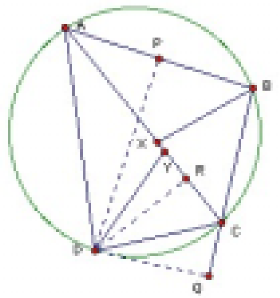

Hence angle APB = 60°. Suppose AD and CF meet at Q. The same argument shows that angle AQF = 60°. So the hexagon angle at A is angle APB + angle AQF = 120°. Similarly for the other angles.

*Finally, note that the only possible configuration is:*



The ratio AB/BC is arbitrary, but the figure is symmetrical under rotations through $120^o$. That follows immediately from either of the two solutions above.

**Problem 4**



**Solution**

APRD is cyclic with diameter AD (because angle APD = angle ARD = $90^o$. Suppose its center is O and its radius r. Angle PAR = ½ angle POR, so PR = 2r sin ½POR = AD sin PAR. Similarly, RQ = CD sin RCQ. (Note that it makes no difference if R, P are on the same or opposite sides of the line AD.) But sin PAR = sin BAC, sin RCQ = sin ACB, so applying the sine rule to the triangle ABC, sin RCQ/sin PAR = AB/BC. Thus we have AD/CD = (PR/RQ) (AB/BC). Suppose the angle bisectors of B, D meet AD at X, Y. Then we have AB/BC = AX/CX and AD/CD = AY/CY. Hence (AY/CY)/(AX/CX) = PR/RQ. So PR = RQ iff X = Y, which is the required result.

*Note that ABCD does not need to be cyclic! Exercise: does it need to be convex?*

**Problem 5**
**Solution** Notice first that if we restrict the sums to $i < j$, then they are halved. The lhs sum is squared and the rhs sum is not, so the the desired inequality with sums restricted to $i < j$ has (1/3) on the rhs instead of (2/3).

Consider the sum of all $|x_i - x_j|$ with $i < j$. $x_1$ occurs in (n-1) terms with a negative sign. $x_2$ occurs in one term with a positive sign and (n-2) terms with a negative sign, and so on. So we get $-(n-1)x_1 - (n-3)x_2 - (n-5)x_3 - ... + (n-1)x_n = \sum (2i-1-n)x_i$.

We can now apply Cauchy-Schwartz. The square of this sum is just $\sum x_i^2 \sum (2i-1-n)^2$.





Looking at the other side of the desired inequality, we see immediately that it is $n \sum x_i^2 - (\sum x_i)^2$. We would like to get rid of the second term, but that is easy because if we add h to every $x_i$ the sums in the desired inequality are unaffected (since they use only differences of $x_i$), so we can choose h so that $\sum x_i$ is zero. Thus we are home if we can show that $\sum (2i-1-n)^2 \le n(n^2 - 1)/3$. That is easy: lhs $= 4 \sum i^2 - 4(n+1) \sum i + n(n+1)^2 = (2/3)n(n+1)(2n+1) - 2n(n+1) + n(n+1)^2 = (1/3)n(n+1)(2(2n+1) - 6 + 3(n+1)) = (1/3)n(n^2 - 1)$ = rhs. That establishes the required inequality.

We have equality iff we have equality at the Cauchy-Schwartz step and hence iff $x_i$ is proportional to (2i-1-n). That implies that $x_{i+1} - x_i$ is constant. So equality implies that the sequence is an AP. But if the sequence is an AP with difference d (so $x_{i+1} = x_i + d$) and we take $x_1 = -(d/2)(n-1)$, then we get $x_i = (d/2)(2i-1-n)$ and $\sum x_i = 0$, so we have equality.

**Problem 6**
**Solution** If p = 2, then we can take q = 3, since squares cannot be 2 mod 3. So suppose p is odd.

Consider $N = 1 + p + p^2 + ... + p^{p-1}$. There are p terms. Since p is odd, that means an odd number of odd terms, so N is odd. Also $N = p + 1 \bmod p^2$, which is not 1 mod $p^2$, so N must have a prime factor q which is not 1 mod $p^2$. We will show that q has the required property.

Since N = 1 mod p, p does not divide N, so q cannot be p. If p = 1 mod q, then N = 1 + 1 + ... + 1 = p mod q. Since N = 0 mod q, that implies q divides p. Contradiction. So q does not divide p - 1.

Now suppose $n^p = p \bmod q$ (*). We have just shown that n cannot be 1 mod q. We have also shown that q is not p, so n cannot be a multiple of q. So assume n is not 0 or 1 mod q. Take the pth power of both sides of (*). Since $(p - 1)N = p^p - 1$, we have $p^p = 1 \bmod q$. So n to the power of $p^2$ is 1 mod q. But $n^{q-1} = 1 \bmod q$ (the well-known Fermat little theorem). So if $d = \gcd(q-1, p^2)$, then $n^d = 1 \bmod q$. We chose q so that q-1 is not divisible by $p^2$, so d must be 1 or p. But we are assuming n is not 1 mod q, so d cannot be 1. So it must be p. In other words, $n^p = 1 \bmod q$. But $n^p = p \bmod q$, so p = 1 mod q. Contradiction (we showed above that q does not divide p - 1).