1959/1: Prove that the fraction $\dfrac{21n+4}{14n+3}$ is irreducible for every natural number n.

1959/2: For what real values of x is

$$\sqrt{x+\sqrt{2x-1}}+\sqrt{x-\sqrt{2x-1}}=A\,,$$

given (a) $A=\sqrt{2}$, (b) $A=1$, (c) $A=2$, where only non-negative real numbers are admitted for square roots?

1959/3: Let a, b, c be real numbers. Consider the quadratic equation in $\cos x$:

$$a\cos^2 x + b\cos x + c = 0\,.$$

Using the numbers a, b, c, form a quadratic equation in $\cos 2x$, whose roots are the same as those of the original equation. Compare the equations in $\cos x$ and $\cos 2x$ for $a=4, b=2, c=-1$.

1959/4: Construct a right triangle with given hypotenuse c such that the median drawn to the hypotenuse is the geometric mean of the two legs of the triangle.

1959/5: An arbitrary point M is selected in the interior of the segment AB. The squares $AMCD$ and $MBEF$ are constructed on the same side of AB, with the segments AM and MB as their respective bases. The circles circumscribed about these squares, with centers P and Q, intersect at M and also at another point N. Let N' denote the point of intersection of the straight lines AF and BC.

(a) Prove that the points N and N' coincide.

(b) Prove that the straight lines MN pass through a fixed point S independent of the choice of M.

(c) Find the locus of the midpoints of the segments PQ as M varies between A and B.

1959/6: Two planes, P and Q, intersect along the line p. The point A is given in the plane P, and the point C in the plane Q; neither of these points lies on the straight line p. Construct an isosceles trapezoid $ABCD$ (with AB parallel to CD) in which a circle can be inscribed, and with vertices B and D lying in the planes P and Q respectively.

1960/1: Determine all three-digit numbers N having the property that N is divisible by 11, and $N/11$ is equal to the sum of the squares of the digits of N.

1960/2: For what values of the variable x does the following inequality hold,

$$\frac{4x^2}{(1-\sqrt{1+2x})^2} < 2x+9\,?$$

1960/3: In a given right triangle ABC, the hypotenuse BC, of length a, is divided into n equal parts (n an odd integer). Let α be the acute angle subtending, from A, that segment which contains the midpoint of the hypotenuse. Let h be the length of the altitude to the hypotenuse of the triangle. Prove:

$$\tan\alpha = \frac{4nh}{(n^2-1)a}\,.$$

1960/4: Construct triangle ABC, given h_a, h_b (the altitudes from A and B) and m_a, the median from vertex A.

1960/5: Consider the cube $ABCDA'B'C'D'$ (with face $ABCD$ directly above face $A'B'C'D'$).

(a) Find the locus of the midpoints of segments XY, where X is any point of AC and Y is any point of $B'D'$.

(b) Find the locus of points Z which lie on the segments XY of part (a) with $ZY = 2XZ$.

1960/6: Consider a cone of revolution with an inscribed sphere tangent to the base of the cone. A cylinder is circumscribed about this sphere so that one of its bases lies in the base of the cone. Let V_1 be the volume of the cone and V_2 the volume of the cylinder.

(a) Prove that $V_1 \neq V_2$.

(b) Find the smallest number k for which $V_1 = kV_2$, for this case, construct the angle subtended by a diameter of the base of the cone at the vertex of the cone.

1960/7: An isosceles trapezoid with bases a and c and altitude h is given.

(a) On the axis of symmetry of this trapezoid, find all points P such that both legs of the trapezoid subtend right angles at P.

(b) Calculate the distance of P from either base.

(c) Determine under what conditions such points P actually exist. (Discuss various cases that might arise.)

1961/1: Solve the system of equations:

$$x+y+z=a\,,\quad x^2+y^2+z^2=b^2\,,\quad xy=z^2\,,$$

where a and b are constants. Give the conditions that a and b must satisfy so that x, y, z (the solutions of the system) are distinct positive numbers.

1961/2: Let a, b, c be the sides of a triangle, and T its area. Prove: $a^2+b^2+c^2 \geq 4\sqrt{3}T$. In what case does equality hold?

1961/3: Solve the equation $\cos^n x - \sin^n x = 1$, where n is a natural number.

1961/4: Consider triangle $P_1P_2P_3$ and a point P within the triangle. Lines P_1P, P_2P, P_3P intersect the opposite sides in points Q_1, Q_2, Q_3 respectively. Prove that, of the numbers $\dfrac{P_1P}{PQ_1}$, $\dfrac{P_2P}{PQ_2}$, $\dfrac{P_3P}{PQ_3}$ at least one is ≤ 2 and at least one is ≥ 2.

1961/5: Construct triangle ABC if $AC = b$, $AB = c$ and $\measuredangle AMB = \omega$, where M is the midpoint of segment BC and $\omega < 90°$. Prove that a solution exists if and only if $b \tan \dfrac{\omega}{2} \leq c < b$. In what case does the equality hold?

1961/6: Consider a plane ε and three non-collinear points A, B, C on the same side of ε; suppose the plane determined by these three points is not parallel to ε. In plane a take three arbitrary points A', B', C'. Let L, M, N be the midpoints of segments AA', BB', CC'; let G be the centroid of triangle LMN.
(We will not consider positions of the points A', B', C' such that the points L, M, N do not form a triangle.)
What is the locus of point G as A', B', C' range independently over the plane ε?

1962/1: Find the smallest natural number n which has the following properties:
(a) Its decimal representation has 6 as the last digit.
(b) If the last digit 6 is erased and placed in front of the remaining digits, the resulting number is four times as large as the original number n.

1962/2: Determine all real numbers x which satisfy the inequality:

$$\sqrt{3-x} - \sqrt{x+1} > \frac{1}{2}.$$

1962/3: Consider the cube $ABCDA'B'C'D'$ ($ABCD$ and $A'B'C'D'$ are the upper and lower bases, respectively, and edges AA', BB', CC', DD' are parallel). The point X moves at constant speed along the perimeter of the square $ABCD$ in the direction $ABCDA$, and the point Y moves at the same rate along the perimeter of the square $B'C'CB$ in the direction $B'C'CBB'$. Points X and Y begin their motion at the same instant from the starting positions A and B', respectively. Determine and draw the locus of the midpoints of the segments XY.

1962/4: Solve the equation $\cos^2 x + \cos^2 2x + \cos^2 3x = 1$.

1962/5: On the circle K there are given three distinct points A, B, C. Construct (using only straightedge and compasses) a fourth point D on K such that a circle can be inscribed in the quadrilateral thus obtained.

1962/6: Consider an isosceles triangle. Let r be the radius of its circumscribed circle and ρ the radius of its inscribed circle. Prove that the distance d between the centers of these two circles is $d = \sqrt{r(r-2\rho)}$.

1962/7: The tetrahedron $SABC$ has the following property: there exist five spheres, each tangent to the edges $SA, SB, SC, BCCA, AB$, or to their extensions.
(a) Prove that the tetrahedron $SABC$ is regular.
(b) Prove conversely that for every regular tetrahedron five such spheres exist.

1963/1: Find all real roots of the equation $\sqrt{x^2-p} + 2\sqrt{x^2-1} = x$, where p is a real parameter.

1963/2: Point A and segment BC are given. Determine the locus of points in space which are vertices of right angles with one side passing through A, and the other side intersecting the segment BC.

1963/3: In an n-gon all of whose interior angles are equal, the lengths of consecutive sides satisfy the relation $a_1 \geq a_2 \geq \cdots \geq a_n$. Prove that $a_1 = a_2 = \cdots = a_n$.

1963/4: Find all solutions x_1, x_2, x_3, x_4, x_5 of the system

$$\begin{aligned} x_5 + x_2 &= yx_1 \\ x_1 + x_3 &= yx_2 \\ x_2 + x_4 &= yx_3 \\ x_3 + x_5 &= yx_4 \\ x_4 + x_1 &= yx_5, \end{aligned}$$

where y is a parameter.

1963/5: Prove that $\cos\dfrac{\pi}{7} - \cos\dfrac{2\pi}{7} + \cos\dfrac{3\pi}{7} = \dfrac{1}{2}$.

1963/6: Five students, A, B, C, D, E, took part in a contest. One prediction was that the contestants would finish in the order $ABCDE$. This prediction was very poor. In fact no contestant finished in the position predicted, and no two contestants predicted to finish consecutively actually did so. A second prediction had the contestants finishing in the order $DAECB$. This prediction was better. Exactly two of the contestants finished in the places predicted, and two disjoint pairs of students predicted to finish consecutively actually did so. Determine the order in which the contestants finished.

1964/1:
(a) Find all positive integers n for which $2^n - 1$ is divisible by 7.
(b) Prove that there is no positive integer n for which $2^n + 1$ is divisible by 7.

1964/2: Suppose a, b, c are the sides of a triangle. Prove that

$$a^2(b+c-a) + b^2(c+a-b) + c^2(a+b-c) \leq 3abc.$$

1964/3: A circle is inscribed in triangle ABC with sides a, b, c. Tangents to the circle parallel to the sides of the triangle are constructed. Each of these tangents cuts off a triangle from ΔABC. In each of these triangles, a circle is inscribed. Find the sum of the areas of all four inscribed circles (in terms of a, b, c).

1964/4: Seventeen people correspond by mail with one another - each one with all the rest. In their letters only three different topics are discussed. Each pair of correspondents deals with only one of these topics. Prove that there are at least three people who write to each other about the same topic.

1964/5: Suppose five points in a plane are situated so that no two of the straight lines joining them are parallel, perpendicular, or coincident. From each point perpendiculars are drawn to all the lines joining the other four points. Determine the maximum number of intersections that these perpendiculars can have.

1964/6: In tetrahedron $ABCD$, vertex D is connected with D_0 the centroid of ΔABC. Lines parallel to DD_0 are drawn through A, B and C. These lines intersect the planes BCD, CAD and ABD in points A_1, B_1 and C_1, respectively. Prove that the volume of $ABCD$ is one third the volume of $A_1B_1C_1D_0$. Is the result true if point D_0 is selected anywhere within ΔABC?

1965/1: Determine all values x in the interval $0 \leq x \leq 2\pi$ which satisfy the inequality $2\cos x \leq \left|\sqrt{1+\sin 2x} - \sqrt{1-\sin 2x}\right| \leq \sqrt{2}$.

1965/2: Consider the system of equations

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 &= 0 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 &= 0 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 &= 0 \end{aligned}$$

with unknowns x_1, x_2, x_3. The coefficients satisfy the conditions:

(a) a_{11}, a_{22}, a_{33} are positive numbers;

(b) the remaining coefficients are negative numbers;

(c) in each equation, the sum of the coefficients is positive.

Prove that the given system has only the solution $x_1 = x_2 = x_3 = 0$.

1965/3: Given the tetrahedron $ABCD$ whose edges AB and CD have lengths a and b respectively. The distance between the skew lines AB and CD is d, and the angle between them is ω. Tetrahedron $ABCD$ is divided into two solids by plane ε, parallel to lines AB and CD. The ratio of the distances of ε from AB and CD is equal to k. Compute the ratio of the volumes of the two solids obtained.

1965/4: Find all sets of four real numbers x_1, x_2, x_3, x_4 such that the sum of any one and the product of the other three is equal to 2.

1965/5: Consider ΔOAB with acute angle AOB. Through a point $M \neq 0$ perpendiculars are drawn to OA and OB, the feet of which are P and Q respectively. The point of intersection of the altitudes of ΔOPQ is H. What is the locus of H if M is permitted to range over

(a) the side AB,

(b) the interior of ΔOAB?

1965/6: In a plane a set of n points ($n \geq 3$) is given. Each pair of points is connected by a segment. Let d be the length of the longest of these segments. We define a diameter of the set to be any connecting segment of length d. Prove that the number of diameters of the given set is at most n.

1966/1: In a mathematical contest, three problems, A, B, C were posed. Among the participants there were 25 students who solved at least one problem each. Of all the contestants who did not solve problem A, the number who solved B was twice the number who solved C. The number of students who solved only problem A was one more than the number of students who solved A and at least one other problem. Of all students who solved just one problem, half did not solve problem A. How many students solved only problem B?

1966/2: Let a, b, c be the lengths of the sides of a triangle, and α, β, γ, respectively, the angles opposite these sides. Prove that if $a + b = \tan\dfrac{\gamma}{2}(a\tan\alpha + b\tan\beta)$, the triangle is isosceles.

1966/3: Prove: The sum of the distances of the vertices of a regular tetrahedron from the center of its circumscribed sphere is less than the sum of the distances of these vertices from any other point in space.

1966/4: Prove that for every natural number n, and for every real number $x \neq k\pi/2^t (t = 0, 1, \ldots, n; k \text{ any integer})$

$$\frac{1}{\sin 2x} + \frac{1}{\sin 4x} + \cdots + \frac{1}{\sin 2^n x} = \cot x - \cot 2^n x.$$

1966/5: Solve the system of equations

$$\begin{array}{lllll} & |a_1 - a_2|\, x_2 & +|a_1 - a_3|\, x_3 & +|a_1 - a_4|\, x_4 & = 1 \\ |a_2 - a_1|\, x_1 & & +|a_2 - a_3|\, x_3 & +|a_2 - a_3|\, x_3 & = 1 \\ |a_3 - a_1|\, x_1 & +|a_3 - a_2|\, x_2 & & & = 1 \\ |a_4 - a_1|\, x_1 & +|a_4 - a_2|\, x_2 & +|a_4 - a_3|\, x_3 & & = 1 \end{array}$$

where a_1, a_2, a_3, a_4 are four different real numbers.

1966/6: In the interior of sides BC, CA, AB of triangle ABC, any points K, L, M, respectively, are selected. Prove that the area of at least one of the triangles AML, BKM, CLK is less than or equal to one quarter of the area of triangle ABC.

1967/1: Let $ABCD$ be a parallelogram with side lengths $AB = a, AD = 1$, and with $\measuredangle BAD = \alpha$. If ΔABD is acute, prove that the four circles of radius 1 with centers A, B, C, D cover the parallelogram if and only if $a \leq \cos\alpha + \sqrt{3}\sin\alpha$.

1967/2: Prove that if one and only one edge of a tetrahedron is greater than 1, then its volume is $\leq 1/8$.

1967/3: Let k, m, n be natural numbers such that $m+k+1$ is a prime greater than $n+1$. Let $c_s = s(s+1)$. Prove that the product

$$(c_{m+1} - c_k)(c_{m+2} - c_k)\ldots(c_{m+n} - c_k)$$

is divisible by the product $c_1 c_2 \ldots c_n$.

1967/4: Let $A_0B_0C_0$ and $A_1B_1C_1$ be any two acute-angled triangles. Consider all triangles ABC that are similar to $\Delta A_1B_1C_1$ (so that vertices A_1, B_1, C_1 correspond to vertices A, B, C, respectively) and circumscribed about triangle $A_0B_0C_0$ (where A_0 lies on BC, B_0 on CA, and AC_0 on AB). Of all such possible triangles, determine the one with maximum area, and construct it.

1967/5: Consider the sequence $\{c_n\}$, where

$$\begin{aligned} c_1 &= a_1 + a_2 + \cdots + a_8 \\ c_2 &= a_1^2 + a_2^2 + \cdots + a_8^2 \\ &\vdots \\ c_n &= a_1^n + a_2^n + \cdots + a_8^n \\ &\vdots \end{aligned}$$

in which $a_1, a_2, \ldots, a_8$ are real numbers not all equal to zero. Suppose that an infinite number of terms of the sequence $\{c_n\}$ are equal to zero. Find all natural numbers n for which $c_n = 0$.

1967/6: In a sports contest, there were m medals awarded on n successive days ($n > 1$). On the first day, one medal and $1/7$ of the remaining $m-1$ medals were awarded. On the second day, two medals and $1/7$ of the now remaining medals were awarded; and so on. On the n-th and last day, the remaining n medals were awarded. How many days did the contest last, and how many medals were awarded altogether?

1968/1: Prove that there is one and only one triangle whose side lengths are consecutive integers, and one of whose angles is twice as large as another.

1968/2: Find all natural numbers x such that the product of their digits (in decimal notation) is equal to $x^2 - 10x - 22$.

1968/3: Consider the system of equations

$$\begin{aligned} ax_1^2 + bx_1 + c &= x_2 \\ ax_2^2 + bx_2 + c &= x_3 \\ &\vdots \\ ax_{n-1}^2 + bx_{n-1} + c &= x_n \\ ax_n^2 + bx_n + c &= x_1, \end{aligned}$$

with unknowns $x_1, x_2, \ldots, x_n$, where a, b, c are real and $a \neq 0$. Let $\Delta = (b-1)^2 - 4ac$. Prove that for this system

(a) if $\Delta < 0$, there is no solution,
(b) if $\Delta = 0$, there is exactly one solution,
(c) if $\Delta > 0$, there is more than one solution.

1968/4: Prove that in every tetrahedron there is a vertex such that the three edges meeting there have lengths which are the sides of a triangle.

1968/5: Let f be a real-valued function defined for all real numbers x such that, for some positive constant a, the equation $f(x+a) = \frac{1}{2} + \sqrt{f(x) - [f(x)]^2}$ holds for all x.

(a) Prove that the function f is periodic (i.e., there exists a positive number b such that $f(x+b) = f(x)$ for all x).
(b) For $a = 1$, give an example of a non-constant function with the required properties.

1968/6: For every natural number n, evaluate the sum

$$\sum_{k=0}^{\infty}\left[\frac{n+2^k}{2^{k+1}}\right] = \left[\frac{n+1}{2}\right] + \left[\frac{n+2}{4}\right] + \cdots + \left[\frac{n+2^k}{2^{k+1}}\right] + \ldots$$

(The symbol $[x]$ denotes the greatest integer not exceeding x.)

1969/1: Prove that there are infinitely many natural numbers a with the following property: the number $z = n^4 + a$ is not prime for any natura1 number n.

1969/2: Let $a_1, a_2, \ldots, a_n$ be real constants, x a real variable, and

$$f(x) = \cos(a_1 + x) + \frac{1}{2}\cos(a_2 + x) + \frac{1}{4}\cos(a_3 + x) + \cdots + \frac{1}{2^{n-1}}\cos(a_n + x).$$

Given that $f(x_1) = f(x_2) = 0$, prove that $x_2 - x_1 = m\pi$ for some integer m.

1969/3: For each value of $k = 1, 2, 3, 4, 5$, find necessary and sufficient conditions on the number $a > 0$ so that there exists a tetrahedron with k edges of length a, and the remaining $6 - k$ edges of length 1.

1969/4: A semicircular arc γ is drawn on AB as diameter. C is a point on γ other than A and B, and D is the foot of the perpendicular from C to AB. We consider three circles, $\gamma_1, \gamma_2, \gamma_3$, all tangent to the line AB. Of these, γ_1 is inscribed in ΔABC, while γ_2 and γ_3 are both tangent to CD and to γ, one on each side of CD. Prove that γ_1, γ_2 and γ_3 have a second tangent in common.

1969/5: Given $n > 4$ points in the plane such that no three are collinear. Prove that there are at least $\binom{n-3}{2}$ convex quadrilaterals whose vertices are four of the given points.

1969/6: Prove that for all real numbers $x_1, x_2, y_1, y_2, z_1, z_2$, with $x_1 > 0$, $x_2 > 0, x_1y_1 - z_1^2 > 0, x_2y_2 - z_2^2 > 0$, the inequality

$$\frac{8}{(x_1+x_2)(y_1+y_2)-(z_1+z_2)^2} \leq \frac{1}{x_1y_1 - z_1^2} + \frac{1}{x_2y_2 - z_2^2}$$

is satisfied. Give necessary and sufficient conditions for equality.

1970/1: Let M be a point on the side AB of ΔABC. Let r_1, r_2 and r be the radii of the inscribed circles of triangles AMC, BMC and ABC. Let q_1, q_2 and q be the radii of the escribed circles of the same triangles that lie in the angle ACB. Prove that $\frac{r_1}{q_1} \cdot \frac{r_2}{q_2} = \frac{r}{q}$.

1970/2: Let a, b and n be integers greater than 1, and let a and b be the bases of two number systems. A_{n-1} and A_n are numbers in the system with base a, and B_{n-1} and $B_n$are numbers in the system with base b; these are related as follows:

$$\begin{aligned} A_n &= x_n x_{n-1} \dots x_0, A_{n-1} = x_{n-1}x_{n-2}\dots x_0, \\ B_n &= x_n x_{n-1} \dots x_0, B_{n-1} = x_{n-1}x_{n-2}\dots x_0, \\ x_n &\neq 0, x_{n-1} \neq 0. \end{aligned}$$

Prove $\frac{A_{n-1}}{A_n} < \frac{B_{n-1}}{B_n}$ if and only if $a > b$.

1970/3: The real numbers $a_0, a_1, \dots, a_n, \dots$ satisfy the condition:

$$1 = a_0 \leq a_1 \leq a_2 \leq \dots \leq a_n \leq \dots .$$

The numbers $b_1, b_2, \dots, b_n, \dots$ are defined by $b_n = \sum_{k=1}^{n} \left(1 - \frac{a_{k-1}}{a_k}\right) \frac{1}{\sqrt{a_k}}$.

(a) Prove that $0 \leq b_n < 2$ for all n.

(b) Given c with $0 \leq c < 2$, prove that there exist numbers $a_0, a_1, \dots$ with the above properties such that $b_n > c$ for large enough n.

1970/4: Find the set of all positive integers n with the property that the set $\{n, n+1, n+2, n+3, n+4, n+5\}$ can be partitioned into two sets such that the product of the numbers in one set equals the product of the numbers in the other set.

1970/5: In the tetrahedron $ABCD$, angle BDC is a right angle. Suppose that the foot H of the perpendicular from D to the plane ABC is the intersection of the altitudes of ΔABC. Prove that

$$(AB + BC + CA)^2 \leq 6(AD^2 + BD^2 + CD^2).$$

For what tetrahedra does equality hold?

1970/6: In a plane there are 100 points, no three of which are collinear. Consider all possible triangles having these points as vertices. Prove that no more than 70% of these triangles are acute-angled.

1971/1: Prove that the following assertion is true for $n = 3$ and $n = 5$, and that it is false for every other natural number $n > 2$:
If $a_1, a_2, \dots, a_n$ are arbitrary real numbers, then

$$(a_1 - a_2)(a_1 - a_3)\dots(a_1 - a_n) + (a_2 - a_1)(a_2 - a_3)\dots(a_2 - a_n) + \dots + (a_n - a_1)(a_n - a_2)\dots(a_n - a_{n-1}) \geq 0$$

1971/2: Consider a convex polyhedron P_1 with nine vertices $A_1A_2, \dots, A_9$; let P_i be the polyhedron obtained from P_1 by a translation that moves vertex A_1 to A_i $(i = 2, 3, \dots, 9)$. Prove that at least two of the polyhedra $P_1, P_2, \dots, P_9$ have an interior point in common.

1971/3: Prove that the set of integers of the form $2^k - 3$ $(k = 2, 3, \dots)$ contains an infinite subset in which every two members are relatively prime.

1971/4: All the faces of tetrahedron $ABCD$ are acute-angled triangles. We consider all closed polygonal paths of the form $XYZTX$ defined as follows: X is a point on edge AB distinct from A and B; similarly, Y, Z, T are interior points of edges $BCCD, DA$, respectively. Prove:

(a) If $\measuredangle DAB + \measuredangle BCD \neq \measuredangle CDA + \measuredangle ABC$, then among the polygonal paths, there is none of minimal length.

(b) If $\measuredangle DAB + \measuredangle BCD = \measuredangle CDA + \measuredangle ABC$, then there are infinitely many shortest polygonal paths, their common length being $2\,AC\sin(\alpha/2)$, where $\alpha = \measuredangle BAC + \measuredangle CAD + \measuredangle DAB$.

1971/5: Prove that for every natural number m, there exists a finite set S of points in a plane with the following property: For every point A in S, there are exactly m points in S which are at unit distance from A.

1971/6: Let $A = (a_{ij})$ $(i, j = 1, 2, \dots, n)$ be a square matrix whose elements are non-negative integers. Suppose that whenever an element $a_{ij} = 0$, the sum of the elements in the ith row and the jth column is $\geq n$. Prove that the sum of all the elements of the matrix is $\geq n^2/2$.

1972/1: Prove that from a set of ten distinct two-digit numbers (in the decimal system), it is possible to select two disjoint subsets whose members have the same sum.

1972/2: Prove that if $n \geq 4$, every quadrilateral that can be inscribed in a circle can be dissected into n quadrilaterals each of which is inscribable in a circle.

1972/3: Let m and n be arbitrary non-negative integers. Prove that

$$\frac{(2m)!(2n)!}{m!n!(m+n)!} \text{ is an integer. } (0! = 1\,.)$$

1972/4: Find all solutions $(x_1, x_2, x_3, x_4, x_5)$ of the system of inequalities

$$\begin{aligned}
(x_1^2 - x_3x_5)(x_2^2 - x_3x_5) &\leq 0 \\
(x_2^2 - x_4x_1)(x_3^2 - x_4x_1) &\leq 0 \\
(x_3^2 - x_5x_2)(x_4^2 - x_5x_2) &\leq 0 \\
(x_4^2 - x_1x_3)(x_5^2 - x_1x_3) &\leq 0 \\
(x_5^2 - x_2x_4)(x_1^2 - x_2x_4) &\leq 0
\end{aligned}$$

where x_1, x_2, x_3, x_4, x_5 are positive real numbers.

1972/5: Let f and g be real-valued functions defined for all real values of x and y, and satisfying the equation

$$f(x+y) + f(x-y) = 2f(x)g(y)\,, \quad \text{for all } x, y\,.$$

Prove that if $f(x)$ is not identically zero, and if $|f(x)| \leq 1$ for all x, then $|g(y)| \leq 1$ for all y.

1972/6: Given four distinct parallel planes, prove that there exists a regular tetrahedron with a vertex on each plane.

1973/1: Point O lies on line g; $\overrightarrow{OP_1}, \overrightarrow{OP_2}, \ldots, \overrightarrow{OP_n}$ are unit vectors such that points $P_1, P_2, \ldots, P_n$ all lie in a plane containing g and on one side of g. Prove that if n is odd, $\left|\overrightarrow{OP_1} + \overrightarrow{OP_2} + \cdots + \overrightarrow{OP_n}\right| \geq 1$. Here $\left|\overrightarrow{OM}\right|$ denotes the length of vector $\overrightarrow{OM}$.

1973/2: Determine whether or not there exists a finite set M of points in space not lying in the same plane such that, for any two points A and B of M, one can select two other points C and D of M so that lines AB and CD are parallel and not coincident.

1973/3: Let a and b be real numbers for which the equation $x^4+ax^3+bx^2+ax+1=0$ has at least one real solution. For all such pairs (a,b), find the minimum value of a^2+b^2.

1973/4: A soldier needs to check on the presence of mines in a region having the shape of an equilateral triangle. The radius of action of his detector is equal to half the altitude of the triangle. The soldier leaves from one vertex of the triangle. What path shouid he follow in order to travel the least possible distance and still accomplish his mission?

1973/5: G is a set of non-constant functions of the real variable x of the form

$$f(x) = ax + b,\ a \text{ and } b \text{ are real numbers,}$$

and G has the following properties:

(a) If f and g are in G, then $g \circ f$ is in G; here $(g \circ f)(x) = g[f(x)]$.
(b) If f is in G, then its inverse f^{-1} is in G; here the inverse of $f(x) = ax + b$ is $f^{-1}(x) = (x-b)/a$.
(c) For every f in G, there exists a real number x_f such that $f(x_f) = x_f$.

Prove that there exists a real number k such that $f(k) = k$ for all f in G.

1973/6: Let $a_1, a_2, \ldots, a_n$ be n positive numbers, and let q be a given real number such that $0 < q < 1$. Find n numbers $b_1, b_2, \ldots, b_n$ for which

(a) $a_k < b_k$ for $k = 1, 2, \ldots, n$,
(b) $q < \dfrac{b_{k+1}}{b_k} < \dfrac{1}{q}$ for $k = 1, 2, \ldots, n-1$,
(c) $b_1 + b_2 + \cdots + b_n < \dfrac{1+q}{1-q}(a_1 + a_2 + \cdots + a_n)$.

1974/1: Three players A, B and C play the following game: On each of three cards an integer is written. These three numbers p, q, r satisfy $0 < p < q < r$. The three cards are shuffled and one is dealt to each player. Each then receives the number of counters indicated by the card he holds. Then the cards are shuffled again; the counters remain with the players.
This process (shuffling, dealing, giving out counters) takes place for at least two rounds. After the last round, A has 20 counters in all, B has 10 and C has 9. At the last round B received r counters. Who received q counters on the first round?

1974/2: In the triangle ABC, prove that there is a point D on side AB such that CD is the geometric mean of AD and DB if and only if $\sin A \sin B \leq \sin^2 \dfrac{C}{2}$.

1974/3: Prove that $\displaystyle\sum_{k=0}^{n} \binom{2n+1}{2k+1} 2^{3k}$ is not divisible by 5 for any integer $n \geq 0$.

1974/4: Consider decompositions of an 8×8 chessboard into p non-overlapping rectangles subject to the following conditions:

(i) Each rectangle has as many white squares as black squares.

(ii) If a_i is the number of white squares in the i-th rectangle, then $a_1 < a_2 < \cdots < a_p$. Find the maximum value of p for which such a decomposition is possible. For this value of p, determine all possible sequences $a_1, a_2, \ldots, a_p$.

1974/5: Determine all possible values of

$$S = \frac{a}{a+b+d} + \frac{b}{a+b+c} + \frac{c}{b+c+d} + \frac{d}{a+c+d}$$

where a, b, c, d are arbitrary positive numbers.

1974/6: Let P be a non-constant polynomial with integer coefficients. If $n(P)$ is the number of distinct integers k such that $(P(k))^2 = 1$, prove that $n(P) - \mathrm{gr}(P) \le 2$, where $\mathrm{gr}(P)$ denotes the degree of the polynomial P.

1975/1: Let x_i, y_i $(i = 1, 2, \ldots, n)$ be real numbers such that

$$x_1 \ge x_2 \ge \cdots \ge x_n \text{ and } y_1 \ge y_2 \ge \cdots \ge y_n .$$

Prove that, if $z_1, z_2, \ldots, z_n$ is any permutation of $y_1, y_2, \ldots, y_n$, then

$$\sum_{i=1}^{n} (x_i - y_i)^2 \le \sum_{i=1}^{n} (x_i - z_i)^2 .$$

1975/2: Let $a_1, a_2, a_3, \ldots$ be an infinite increasing sequence of positive integers. Prove that for every $p \ge 1$ there are infinitely many a_m which can be written in the form $a_m = xa_p + ya_q$ with x, y positive integers and $q > p$.

1975/3: On the sides of an arbitrary triangle ABC, triangles ABR, BCP, CAQ are constructed externally with $\measuredangle CBP = \measuredangle CAQ = 45°, \measuredangle BCP = \measuredangle ACQ = 30°, \measuredangle ABR = \measuredangle BAR = 15°$. Prove that $\measuredangle QRP = 90°$ and $QR = RP$.

1975/4: When 4444^{4444} is written in decimal notation, the sum of its digits is A. Let B be the sum of the digits of A. Find the sum of the digits of B. (A and B are written in decimal notation.)

1975/5: Determine, with proof, whether or not one can find 1975 points on the circumference of a circle with unit radius such that the distance between any two of them is a rational number.

1975/6: Find all polynomials P, in two variables, with the following properties:

(i) for a positive integer n and all real t, x, y $P(tx, ty) = t^n P(x, y)$ (that is, P is homogeneous of degree n),

(ii) for all real a, b, c, $P(b + c, a) + P(c + a, b) + P(a + b, c) = 0$,

(iii) $P(1, 0) = 1$.

1976/1: In a plane convex quadrilateral of area 32, the sum of the lengths of two opposite sides and one diagonal is 16. Determine all possible lengths of the other diagonal.

1976/2: Let $P_1(x) = x^2 - 2$ and $P_j(x) = P_1(P_{j-1}(x))$ for $j = 2, 3, \ldots$. Show that, for any positive integer n, the roots of the equation $P_n(x) = x$ are real and distinct.

1976/3: A rectangular box can be filled completely with unit cubes. If one places as many cubes as possible, each with volume 2, in the box, so that their edges are parallel to the edges of the box, one can fill exactly 40% of the box. Determine the possible dimensions of all such boxes.

1976/4: Determine, with proof, the largest number which is the product of positive integers whose sum is 1976.

1976/5: Consider the system of p equations in $q = 2p$ unknowns $x_1, x_2, \ldots, x_q$:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1q}x_q &= 0 \\ &\vdots \\ a_{p1}x_1 + a_{p2}x_2 + \cdots + a_{pq}x_q &= 0 \end{aligned}$$

with every coefficient a_{ij} member of the set $\{-1, 0, 1\}$. Prove that the system has a solution $(x_1, x_2, \ldots, x_q)$ such that

(a) all x_j $(j = 1, 2, \ldots, q)$ are integers,

(b) there is at least one value of j for which $x_j \ne 0$,

(c) $|x_j| \le q (j = 1, 2, \ldots, q)$.

1976/6: A sequence $\{u_n\}$ is defined by

$$u_0 = 2, u_1 = 5/2, u_{n+1} = u_n(u_{n-1}^2 - 2) - u_1 \text{ for } n = 1, 2, \ldots$$

Prove that for positive integers n, $[u_n] = 2^{[2^n - (-1)^n]/3}$ where $[x]$ denotes the greatest integer $\le x$.

1977/1: Equilateral triangles ABK, BCL, CDM, DAN are constructed inside the square $ABCD$. Prove that the midpoints of the four segments KL, LM, MN, NK and the midpoints of the eight segments $AKBK, BL, CL, CM, DM, DN, AN$ are the twelve vertices of a regular dodecagon.

1977/2: In a finite sequence of real numbers the sum of any seven successive terms is negative, and the sum of any eleven successive terms is positive. Determine the maximum number of terms in the sequence.

1977/3: Let n be a given integer > 2, and let V_n be the set of integers $1 + kn$, where $k = 1, 2, \ldots$. A number $m \in V_n$ is called *indecomposable* in V_n if there do not exist numbers $p, q \in V_n$ such that $pq = m$. Prove that there exists a number $r \in V_n$ that can be expressed as the product of elements indecomposable in V_n in more than one way. (Products which differ only in the order of their factors will be considered the same.)

1977/4: Four real constants a, b, A, B are given, and

$$f(\theta) = 1 - a\cos\theta - b\sin\theta - A\cos 2\theta - B\sin 2\theta.$$

Prove that if $f(\theta) \geq 0$ for all real θ, then $a^2 + b^2 \leq 2$ and $A^2 + B^2 \leq 1$.

1977/5: Let a and b be positive integers. When $a^2 + b^2$ is divided by $a + b$, the quotient is q and the remainder is r. Find all pairs (a, b) such that $q^2 + r = 1977$.

1977/6: Let $f(n)$ be a function defined on the set of all positive integers and having all its values in the same set. Prove that if $f(n+1) > f(f(n))$ for each positive integer n, then $f(n) = n$ for each n.

1978/1: m and n are natural numbers with $1 \leq m < n$. In their decimal representations, the last three digits of 1978^m are equal, respectively, to the last three digits of 1978^n. Find m and n such that $m + n$ has its least value.

1978/2: P is a given point inside a given sphere. Three mutually perpendicular rays from P intersect the sphere at points U, V, and $W; Q$ denotes the vertex diagonally opposite to P in the parallelepiped determined by PU, PV, and PW. Find the locus of Q for all such triads of rays from P

1978/3: The set of all positive integers is the union of two disjoint subsets

$$\{f(1), f(2), \ldots, f(n), \ldots\},\ \{g(1), g(2), \ldots, g(n), \ldots\},$$

where

$$f(1) < f(2) < \cdots < f(n) < \ldots,\quad g(1) < g(2) < \cdots < g(n) < \ldots,$$

and $g(n) = f(f(n)) + 1$ for all $n \geq 1$. Determine $f(240)$.

1978/4: In triangle $ABC, AB = AC$. A circle is tangent internally to the circumcircle of triangle ABC and also to sides AB, AC at P, Q, respectively. Prove that the midpoint of segment PQ is the center of the incircle of triangle ABC.

1978/5: Let $\{a_k\}(k = 1, 2, 3, \ldots, n, \ldots)$ be a sequence of distinct positive integers. Prove that for all natural numbers n, $\displaystyle\sum_{k=1}^{n} \frac{a_k}{k^2} \geq \sum_{k=1}^{n} \frac{1}{k}$.

1978/6: An international society has its members from six different countries. The list of members contains 1978 names, numbered $1, 2, \ldots, 1978$. Prove that there is at least one member whose number is the sum of the numbers of two members from his own country, or twice as large as the number of one member from his own country.

1979/1: Let p and q be natural numbers such that

$$\frac{p}{q} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots - \frac{1}{1318} + \frac{1}{1319}.$$

Prove that p is divisible by 1979.

1979/2: A prism with pentagons $A_1A_2A_3A_4A_5$ and $B_1B_2B_3B_4B_5$ as top and bottom faces is given. Each side of the two pentagons and each of the line-segments A_iB_j for all $i, j = 1, \ldots, 5$, is colored either red or green. Every triangle whose vertices are vertices of the prism and whose sides have all been colored has two sides of a different color. Show that all 10 sides of the top and bottom faces are the same color.

1979/3: Two circles in a plane intersect. Let A be one of the points of intersection. Starting simultaneously from A two points move with constant speeds, each point travelling along its own circle in the same sense. The two points return to A simultaneously after one revolution. Prove that there is a fixed point P in the plane such that, at any time, the distances from P to the moving points are equal.

1979/4: Given a plane π, a point P in this plane and a point Q not in π, find all points R in π such that the ratio $(QP + PA)/QR$ is a maximum.

1979/5: Find all real numbers a for which there exist non-negative real numbers x_1, x_2, x_3, x_4, x_5 satisfying the relations

$$\sum_{k=1}^{5} kx_k = a, \sum_{k=1}^{5} k^3x_k = a^2, \sum_{k=1}^{5} k^5x_k = a^3.$$

1979/6: Let A and E be opposite vertices of a regular octagon. A frog starts jumping at vertex A. From any vertex of the octagon except E, it may jump to either of the two adjacent vertices. When it reaches vertex E, the frog stops and stays there. Let a_n be the number of distinct paths of exactly n jumps ending at E. Prove that $a_{2n-1} = 0$,

$$a_{2n} = \frac{1}{\sqrt{2}}(x^{n-1} - y^{n-1}),\quad n = 1, 2, 3, \ldots,$$

where $x = 2 + \sqrt{2}$ and $y = 2 - \sqrt{2}$.

Note. A path of n jumps is a sequence of vertices $(P_0, \ldots, P_n)$ such that
(i) $P_0 = A, P_n = E$;
(ii) for every $i, 0 \leq i \leq n-1, P_i$ is distinct from E;
(iii) for every $i, 0 \leq i \leq n-1, P_i$ and P_{i+1} are adjacent.

1981/1: P is a point inside a given triangle $ABC.D, E, F$ are the feet of the perpendiculars from P to the lines BC, CA, AB respectively. Find all P for which

$$\frac{BC}{PD} + \frac{CA}{PE} + \frac{AB}{PF} \text{ is least}.$$

1981/2: Let $1 \le r \le n$ and consider all subsets of r elements of the set $\{1, 2, \ldots, n\}$. Each of these subsets has a smallest member. Let $F(n, r)$ denote the arithmetic mean of these smallest numbers; prove that $F(n, r) = \dfrac{n+1}{r+1}$.

1981/3: Determine the maximum value of $m^3 + n^3$,where m and n are integers satisfying $m, n \in \{1, 2, \ldots, 1981\}$ and $(n^2 - mn - m^2)^2 = 1$.

1981/4:

(a) For which values of $n > 2$ is there a set of n consecutive positive integers such that the largest number in the set is a divisor of the least common multiple of the remaining $n - 1$ numbers?

(b) For which values of $n > 2$ is there exactly one set having the stated property?

1981/5: Three congruent circles have a common point O and lie inside a given triangle. Each circle touches a pair of sides of the triangle. Prove that the incenter and the circumcenter of the triangle and the point O are collinear.

1981/6: The function $f(x, y)$ satisfies (1) $f(0, y) = y + 1$, (2) $f(x + 1, 0) = f(x, 1)$, (3) $f(x+1, y+1) = f(x, f(x+1, y))$, for all non-negative integers x, y. Determine $f(4, 1981)$.

1982/1: The function $f(n)$ is defined for all positive integers n and takes on non-negative integer values. Also, for all m, n

$$f(m + n) - f(m) - f(n) = 0 \text{ or } 1$$
$$f(2) = 0, f(3) > 0, \text{ and } f(9999) = 3333.$$

Determine $f(1982)$.

1982/2: A non-isosceles triangle $A_1A_2A_3$ is given with sides a_1, a_2, a_3 (a_i is the side opposite A_i). For all $i = 1, 2, 3, M_i$ is the midpoint of side a_i, and T_i. is the point where the incircle touches side a_i. Denote by S_i the reflection of T_i in the interior bisector of angle A_i. Prove that the lines M_1, S_1, M_2S_2, and M_3S_3 are concurrent.

1982/3: Consider the infinite sequences $\{x_n\}$ of positive real numbers with the properties, $x_0 = 1$, and for all $i \ge 0$, $x_{i+1} \le x_i$.

(a) Prove that for every such sequence, there is an $n \ge 1$ such that

$$\frac{x_0^2}{x_1} + \frac{x_1^2}{x_2} + \cdots + \frac{x_{n-1}^2}{x_n} \ge 3.999.$$

(b) Find such a sequence for which

$$\frac{x_0^2}{x_1} + \frac{x_1^2}{x_2} + \cdots + \frac{x_{n-1}^2}{x_n} < 4.$$

1982/4: Prove that if n is a positive integer such that the equation

$$x^3 - 3xy^2 + y^3 = n$$

has a solution in integers (x, y), then it has at least three such solutions.

Show that the equation has no solutions in integers when $n = 2891$.

1982/5: The diagonals AC and CE of the regular hexagon $ABCDEF$ are divided by the inner points M and N, respectively, so that $\dfrac{AM}{AC} = \dfrac{CN}{CE} = r$. Determine r if B, M, and N are collinear.

1982/6: Let S be a square with sides of length 100, and let L be a path within S which does not meet itself and which is composed of line segments $A_0A_1, A_1A_2, \ldots,$ $A_{n-1}A_n$ with $A_0 \ne A_n$. Suppose that for every point P of the boundary of S there is a point of L at a distance from P not greater than $1/2$. Prove that there are two points X and Y in L such that the distance between X and Y is not greater than 1, and the length of that part of L which lies between X and Y is not smaller than 198.

1983/1: Find all functions f defined on the set of positive real numbers which take positive real values and satisfy the conditions:

(i) $f(xf(y)) = yf(x)$ for all positive x, y;

(ii) $f(x) \to 0$ as $x \to \infty$.

1983/2: Let A be one of the two distinct points of intersection of two unequal coplanar circles C_1 and C_2 with centers O_1 and O_2, respectively. One of the common tangents to the circles touches C_1 at P_1 and C_2 at P_2, while the other touches C_1 at Q_1 and C_2 at Q_2. Let M_1 be the midpoint of P_1Q_1,and M_2 be the midpoint of P_2Q_2. Prove that $\angle O_1AO_2 = \angle M_1AM_2$.

1983/3: Let a, b and c be positive integers, no two of which have a common divisor greater than 1. Show that $2abc - ab - bc - ca$ is the largest integer which cannot be expressed in the form $xbc + yca + zab$,where x, y and z are non-negative integers.

1983/4: Let ABC be an equilateral triangle and $\mathcal{E}$ the set of all points contained in the three segments AB, BC and CA (including A, B and C). Determine whether, for every partition of $\mathcal{E}$ into two disjoint subsets, at least one of the two subsets contains the vertices of a right-angled triangle. Justify your answer.

1983/5: Is it possible to choose 1983 distinct positive integers, all less than or equal to 10^5, no three of which are consecutive terms of an arithmetic progression? Justify your answer.

1983/6: Let a, b and c be the lengths of the sides of a triangle. Prove that

$$a^2b(a-b)+b^2c(b-c)+c^2a(c-a)\geq 0\,.$$

Determine when equality occurs.

1984/1: Prove that $0 \leq yz+zx+xy-2xyz \leq 7/27$, where x, y and z are non-negative real numbers for which $x+y+z=1$.

1984/2: Find one pair of positive integers a and b such that:

(i) $ab(a+b)$ is not divisible by 7;

(ii) $(a+b)^7-a^7-b^7$ is divisible by 7^7.

Justify your answer.

1984/3: In the plane two different points O and A are given. For each point X of the plane, other than O, denote by $a(X)$ the measure of the angle between OA and OX in radians, counterclockwise from $OA (0 \leq a(X) < 2\pi)$. Let $C(X)$ be the circle with center O and radius of length $OX + a(X)/OX$. Each point of the plane is colored by one of a finite number of colors. Prove that there exists a point Y for which $a(Y) > 0$ such that its color appears on the circumference of the circle $C(Y)$.

1984/4: Let $ABCD$ be a convex quadrilateral such that the line CD is a tangent to the circle on AB as diameter. Prove that the line AB is a tangent to the circle on CD as diameter if and only if the lines BC and AD are parallel.

1984/5: Let d be the sum of the lengths of all the diagonals of a plane convex polygon with n vertices ($n>3$), and let p be its perimeter. Prove that

$$n-3<\frac{2d}{p}<\left[\frac{n}{2}\right]\left[\frac{n+1}{2}\right]-2,$$

where $[x]$ denotes the greatest integer not exceeding x.

1984/6: Let a, b, c and d be odd integers such that $0<a<b<c<d$ and $ad=bc$. Prove that if $a+d=2^k$ and $b+c=2^m$ for some integers k and m, then $a=1$.

1985/1: A circle has center on the side AB of the cyclic quadrilateral $ABCD$. The other three sides are tangent to the circle. Prove that $AD+BC=AB$.

1985/2: Let n and k be given relatively prime natural numbers, $k<n$. Each number in the set $M=\{1,2,\ldots,n-1\}$ is colored either blue or white. It is given that:

(i) for each $i \in M$, both i and $n-i$ have the same color;

(ii) for each $i \in M$, $i \neq k$, both i and $|i-k|$ have the same color.

Prove that all numbers in M must have the same color.

1985/3: For any polynomial $P(x)=a_0+a_1x+\cdots+a_kx^k$ with integer coefficients, the number of coefficients which are odd is denoted by $w(P)$. For $i=0,1,\ldots$, let $Q_i(x)=(1+x)^i$. Prove that if $i_1i_2,\ldots,i_n$ are integers such that $0\leq i_1<i_2<\cdots<i_n$, then $w(Q_{i_1}+Q_{i_2},++Q_{i_n})\geq w(Q_{i_1})$.

1985/4: Given a set M of 1985 distinct positive integers, none of which has a prime divisor greater than 26. Prove that M contains at least one subset of four distinct elements whose product is the fourth power of an integer.

1985/5: A circle with center O passes through the vertices A and C of triangle ABC and intersects the segments AB and BC again at distinct points K and N, respectively. The circumscribed circles of the triangles ABC and EBN intersect at exactly two distinct points B and M. Prove that angle OMB is a right angle.

1985/6: For every real number x_1, construct the sequence $x_1, x_2, \ldots$ by setting

$$x_{n+1}=x_n\left(x_n+\frac{1}{n}\right)\quad\text{for each } n\geq 1.$$

Prove that there exists exactly one value of x_1 for which $0<x_n<x_{n+1}<1$ for every n.

1986/1: Let d be any positive integer not equal to 2, 5, or 13. Show that one can find distinct a, b in the set $\{2,5,13,d\}$ such that $ab-1$ is not a perfect square.

1986/2: A triangle $A_1A_2A_3$ and a point P_0 are given in the plane. We define $A_s=A_{s-3}$ for all $s\geq 4$. We construct a set of points P_1, P_2, P_3, …, such that P_{k+1} is the image of P_k under a rotation with center A_{k+1} through angle 120° clockwise (for $k=0,1,2,\ldots$). Prove that if $P_{1986}=P_0$, then the triangle $A_1A_2A_3$ is equilateral.

1986/3: To each vertex of a regular pentagon an integer is assigned in such a way that the sum of all five numbers is positive. If three consecutive vertices are assigned the numbers x, y, z respectively and $y<0$ then the following operation is allowed: the numbers x, y, z are replaced by $x+y$, $-y$, $z+y$ respectively. Such an operation is performed repeatedly as long as at least one of the five numbers is negative. Determine whether this procedure necessarily comes to and end after a finite number of steps.

1986/4: Let A, B be adjacent vertices of a regular n-gon ($n\geq 5$) in the plane having center at O. A triangle XYZ, which is congruent to and initially conincides with OAB, moves in the plane in such a way that Y and Z each trace out the whole boundary of the polygon, X remaining inside the polygon. Find the locus of X.

1986/5: Find all functions f, defined on the non-negative real numbers and taking non-negative real values, such that:
(i) $f(xf(y))f(y) = f(x + y)$ for all x, $y \geq 0$,
(ii) $f(2) = 0$,
(iii) $f(x) \neq 0$ for $0 \leq x < 2$.

1986/6: One is given a finite set of points in the plane, each point having integer coordinates. Is it always possible to color some of the points in the set red and the remaining points white in such a way that for any straight line L parallel to either one of the coordinate axes the difference (in absolute value) between the numbers of white point and red points on L is not greater than 1?

1987/1: Let $p_n(k)$ be the number of permutations of the set $\{1, \ldots, n\}$, $n \geq 1$, which have exactly k fixed points. Prove that $\displaystyle\sum_{k=0}^{n} k \cdot p_n(k) = n!$.
(**Remark:** A permutation f of a set S is a one-to-one mapping of S onto itself. An element i in S is called a fixed point of the permutation f if $f(i) = i$.)

1987/2: In an acute-angled triangle ABC the interior bisector of the angle A intersects BC at L and intersects the circumcircle of ABC again at N. From point L perpendiculars are drawn to AB and AC, the feet of these perpendiculars being K and M respectively. Prove that the quadrilateral $AKNM$ and the triangle ABC have equal areas.

1987/3: Let x_1, x_2, …, x_n be real numbers satisfying $x_1^2 + x_2^2 + \cdots + x_n^2 = 1$. Prove that for every integer $k \geq 2$ there are integers a_1, a_2, …, a_n, not all 0, such that $|a_i| \leq k - 1$ for all i and

$$|a_1 x_1 + a_1 x_2 + \cdots + a_n x_n| \leq \frac{(k-1)\sqrt{n}}{k^n - 1}.$$

1987/4: Prove that there is no function f from the set of non-negative integers into itself such that $f(f(n)) = n + 1987$ for every n.

1987/5: Let n be an integer greater than or equal to 3. Prove that there is a set of n points in the plane such that the distance between any two points is irrational and each set of three points determines a non-degenerate triangle with rational area.

1987/6: Let n be an integer greater than or equal to 2. Prove that if $k^2 + k + n$ is prime for all integers k such that $0 \leq k \leq \sqrt{n/3}$, then $k^2 + k + n$ is prime for all integers k such that $0 \leq k \leq n - 2$.

1988/1: Consider two coplanar circles of radii R and r ($R > r$) with the same center. Let P be a fixed point on the smaller circle and B a variable point on the larger circle. The line BP meets the larger circle again at C. The perpendicular l to BP at P meets the smaller circle again at A. (If l is tangent to the circle at P then $A = P$.)
(i) Find the set of values of $BC^2 + CA^2 + AB^2$.
(ii) Find the locus of the midpoint of BC.

1988/2: Let n be a positive integer and let A_1, A_2, …, A_{2n+1} be subsets of a set B. Suppose that:
(a) Each A_i has exactly $2n$ elements,
(b) Each $A_i \cap A_j$ $(1 \leq i < j \leq 2n + 1)$ contains exactly one element, and
(c) Every element of B belongs to at least two of the A_i.
For which values of n can one assign to every element of B one of the numbers 0 and 1 in such a way that A_i has 0 assigned to exactly n of its elements?

1988/3: A function f is defined on the positive integers by

$$\begin{aligned} f(1) &= 1, \quad f(3) = 3, \\ f(2n) &= f(n), \\ f(4n+1) &= 2f(2n+1) - f(n), \\ f(4n+3) &= 3f(2n+1) - 2f(n), \end{aligned}$$

for all positive integers n.
Determine the number of positive integers n, less than or equal to 1988, for which $f(n) = n$.

1988/4: Show that set of real numbers x which satisfy the inequality

$$\sum_{k=1}^{70} \frac{k}{x-k} \geq \frac{5}{4}$$

is a union of disjoint intervals, the sum of whose lengths is 1988.

1988/5: ABC is a triangle right-angled at A, and D is the foot of the altitude from A. The straight line joining the incenters of the triangles ABD, ACD intersects the sides AB, AC at the points K, L respectively. S and T denote the areas of the triangles ABC and AKL respectively. Show that $S \geq 2T$.

1988/6: Let a and b be positive integers such that $ab + 1$ divides $a^2 + b^2$. Show that

$$\frac{a^2 + b^2}{ab + 1} \text{ is the square of an integer.}$$

1989/1: Prove that the set $\{1, 2, \ldots, 1989\}$ can be expressed as the disjoint union of subsets A_i $(i = 1, 2, \ldots, 117)$ such that:

(i) Each A_i contains 17 elements;
(ii) The sum of all the elements in each A_i is the same.

1989/2: In an acute-angled triangle ABC the internal bisector of angle A meets the circumcircle of the triangle again at A_1. Points B_1 and C_1 are defined similarly. Let A_0 be the point of intersection of the line AA_1 with the external bisectors of angles B and C. Points B_0 and C_0 are defined similarly. Prove that:
(i) The area of the triangle $A_0B_0C_0$ is twice the area of the hexagon $AC_1BA_1CB_1$.
(ii) The area of the triangle $A_0B_0C_0$ is at least four times the area of the triangle ABC.

1989/3: Let n and k be positive integers and let S be a set of n points in the plane such that
(i) No three points of S are collinear, and
(ii) For any point P of S there are at least k points of S equidistant from P.
Prove that $k < \frac{1}{2} + \sqrt{2n}$.

1989/4: Let $ABCD$ be a convex quadrilateral such that the sides AB, AD, BC satisfy $AB = AD + BC$. There exists a point P inside the quadrilateral at a distance h from the line CD such that $AP = h + AD$ and $BP = h + BC$. Show that $\frac{1}{\sqrt{h}} \geq \frac{1}{\sqrt{AD}} + \frac{1}{\sqrt{BC}}$.

1989/5: Prove that for each positive integer n there exist n consecutive positive integers none of which is an integral power of a prime number.

1989/6: A permutation $(x_1, x_2, \ldots, x_m)$ of the set $\{1, 2, \ldots, 2n\}$, where n is a positive integer, is said to have property P if $|x_i - x_{i+1}| = n$ for at least one i in $\{1, 2, \ldots, 2n-1\}$. Show that, for each n, there are more permutations with property P than without.

1990/1: Chords AB and CD of a circle intersect at a point E inside the circle. Let M be an interior point of the segment EB. The tangent line at E to the circle through D, E, and M intersects the lines BC and AC at F and G, respectively. If $\frac{AM}{AB} = t$, find $\frac{EG}{EF}$ in terms of t.

1990/2: Let $n \geq 3$ and consider a set E of $2n-1$ distinct points on a circle. Suppose that exactly k of these points are to be colored black. Such a coloring is "good" if there is at least one pair of black points such that the interior of one of the arcs between them contains exactly n points from E. Find the smallest value of k so that every such coloring of k points of E is good.

1990/3: Determine all integers $n > 1$ such that $\frac{2^n + 1}{n^2}$ is an integer.

1990/4: Let $\mathbb{Q}^+$ be the set of positive rational numbers. Construct a function $f : \mathbb{Q}^+ \to \mathbb{Q}^+$ such that $f(xf(y)) = \frac{f(x)}{y}$ for all x, y in $\mathbb{Q}^+$.

1990/5: Given an initial integer $n_0 > 1$, two players, $\mathcal{A}$ and $\mathcal{B}$, choose integers n_1, n_2, n_3, ... alternately according to the following rules:
Knowing n_{2k}, $\mathcal{A}$ chooses any integer n_{2k+1} such that $n_{2k} \leq n_{2k+1} \leq n_{2k}^2$.
Knowing n_{2k+1}, $\mathcal{B}$ chooses any integer n_{2k+2} such that n_{2k+1}/n_{2k+2} is a prime raised to a positive integer power.
Player $\mathcal{A}$ wins the game by choosing the number 1990; player $\mathcal{B}$ wins by choosing the number 1. For which n_0 does:
(a) $\mathcal{A}$ have a winning strategy?
(b) $\mathcal{B}$ have a winning strategy?
(c) Neither player have a winning strategy?

1990/6: Prove that there exists a convex 1990-gon with the following two properties:
(a) All angles are equal.
(b) The lengths of the 1990 sides are the numbers 1^2, 2^2, 3^2, ..., 1990^2 in some order.

1991/1: Given a triangle ABC, let I be the center of its inscribed circle. The internal bisectors of the angles A, B, C meet the opposite sides in A', B', C' respectively. Prove that $\frac{1}{4} < \frac{AI \cdot BI \cdot CI}{AA' \cdot BB' \cdot CC'} \leq \frac{8}{27}$.

1991/2: Let $n > 6$ be an integer and $a_1, a_2, \ldots, a_k$ be all the natural numbers less than n and relatively prime to n. If $a_2 - a_1 = a_3 - a_2 = \cdots = a_k - a_{k-1} > 0$, prove that n must be either a prime number or a power of 2.

1991/3: Let $S = \{1, 2, 3, \ldots, 280\}$. Find the smallest integer n such that each n-element subset of S contains five numbers which are pairwise relatively prime.

1991/4: Suppose G is a connected graph with k edges. Prove that it is possible to label the edges $1, 2, \ldots, k$ in such a way that at each vertex which belongs to two or more edges, the greatest common divisor of the integers labeling those edges is equal to 1.
[A *graph* consists of a set of points, called *vertices*, together with a set of *edges* joining certain pairs of distinct vertices. Each pair of vertices u, v belongs to at most one edge. The graph G is *connected* if for each pair of distinct vertices x, y there is some sequence of vertices $x = v_0, v_1, v_2, \ldots, v_m = y$ such that each pair v_i, v_{i+1} $(0 \leq i < m)$ is joined by an edge of G.]

1991/5: Let ABC be a triangle and P an interior point of ABC. Show that at least one of the angles $\angle PAB$, $\angle PBC$, $\angle PCA$ is less than or equal to $30°$.

1991/6: An infinite sequence $x_0, x_1, x_2, \ldots$ of real numbers is said to be *bounded* if there is a constant C such that $|x_i| \leq C$ for every $i \geq 0$.
Given any real number $a > 1$, construct a bounded infinite sequence $x_0, x_1, x_2, \ldots$ such that $|x_i - x_j||i - j|^a \geq 1$ for every pair of distinct nonnegative integers i, j.

1992/1: Find all integers a, b, c with $1 < a < b < c$ such that $(a-1)(b-1)(c-1)$ is a divisor of $abc - 1$.

1992/2: Let $\mathbb{R}$ denote the set of all real numbers. Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that $f\left(x^2 + f(y)\right) = y + (f(x))^2 \quad \text{for all}\, x, y \in \mathbb{R}$.

1992/3: Consider nine points in space, no four of which are coplanar. Each pair of points is joined by an edge (that is, a line segment) and each edge is either colored blue or red or left uncolored. Find the smallest value of n such that whenever exactly n edges are colored, the set of colored edges necessarily contains a triangle all of whose edges have the same color.

1992/4: In the plane let C be a circle, L a line tangent to the circle C, and M a point on L. Find the locus of all points P with the following property: there exists two points Q, R on L such that M is the midpoint of QR and C is the inscribed circle of triangle PQR.

1992/5: Let S be a finite set of points in three-dimensional space. Let S_x, S_y, S_z be the sets consisting of the orthogonal projections of the points of S onto the yz-plane, zx-plane, xy-plane, respectively. Prove that $|S|^2 \leq |S_x| \cdot |S_y| \cdot |S_z|$, where $|A|$ denotes the number of elements in the finite set $|A|$. (**Note:** The orthogonal projection of a point onto a plane is the foot of the perpendicular from that point to the plane.)

1992/6: For each positive integer n, $S(n)$ is defined to be the greatest integer such that, for every positive integer $k \leq S(n)$, n^2 can be written as the sum of k positive squares.

(a) Prove that $S(n) \leq n^2 - 14$ for each $n \geq 4$.
(b) Find an integer n such that $S(n) = n^2 - 14$.
(c) Prove that there are infintely many integers n such that $S(n) = n^2 - 14$.

1993/1: Let $f(x) = x^n + 5x^{n-1} + 3$, where $n > 1$ is an integer. Prove that $f(x)$ cannot be expressed as the product of two nonconstant polynomials with integer coefficients.

1993/2: Let D be a point inside acute triangle ABC such that $\measuredangle ADB = \measuredangle ACB + \pi/2$ and $AC \cdot BD = AD \cdot BC$.

(a) Calculate the ratio $(AB \cdot CD)/(AC \cdot BD)$.
(b) Prove that the tangents at C to the circumcircles of $\triangle ACD$ and $\triangle BCD$ are perpendicular.

1993/3: On an infinite chessboard, a game is played as follows. At the start, n^2 pieces are arranged on the chessboard in an n by n block of adjoining squares, one piece in each square. A move in the game is a jump in a horizontal or vertical direction over an adjacent occupied square to an unoccupied square immediately beyond. The piece which has been jumped over is removed.

Find those values of n for which the game can end with only one piece remaining on the board.

1993/4: For three points P, Q, R in the plane, we define $m(PQR)$ as the minimum length of the three altitudes of $\triangle PQR$.
(If the points are collinear, we set $m(PQR) = 0$.)

Prove that for points A, B, C, X in the plane, $m(ABC) \leq m(ABX) + m(AXC) + m(XBC)$.

1993/5: Does there exist a function $f : \mathbf{N} \to \mathbf{N}$ such that $f(1) = 2$, $f(f(n)) = f(n) + n$ for all $n \in \mathbf{N}$, and $f(n) < f(n+1)$ for all $n \in \mathbf{N}$?

1993/6: There are n lamps $L_0, \ldots, L_{n-1}$ in a circle ($n > 1$), where we denote $L_{n+k} = L_k$. (A lamp at all times is either on or off.) Perform steps $s_0, s_1, \ldots$ as follows: at step s_i, if L_{i-1} is lit, switch L_i from on to off or vice versa, otherwise do nothing. Initially all lamps are on. Show that:

(a) There is a positive integer $M(n)$ such that after $M(n)$ steps all the lamps are on again;
(b) If $n = 2^k$, we can take $M(n) = n^2 - 1$;
(c) If $n = 2^k + 1$, we can take $M(n) = n^2 - n + 1$.

1994/1: Let m and n be positive integers. Let $a_1, a_2, \ldots, a_m$ be distinct elements of $\{1, 2, \ldots, n\}$ such that whenever $a_i + a_j \leq n$ for some i, j, $1 \leq i \leq j \leq m$, there exists k, $1 \leq k \leq m$, with $a_i + a_j = a_k$. Prove that

$$\frac{a_1 + a_2 + \cdots + a_m}{m} \geq \frac{n+1}{2}.$$

1994/2: ABC is an isosceles triangle with $AB = AC$. Suppose that

(a) M is the midpoint of BC and O is the point on the line AM such that OB is perpendicular to AB;
(b) Q is an arbitrary point on the segment BC different from B and C;
(c) E lies on the line AB and F lies on the line AC such that E, Q, F are distinct and collinear.

Prove that OQ is perpendicular to EF if and only if $QE = QF$.

1994/3: For any positive integer k, let $f(k)$ be the number of elements in the set $\{k+1, k+2, \dots, 2k\}$ whose base 2 representation has precisely three 1s.

(a) Prove that, for each positive integer m, there exists at least one positive integer k such that $f(k) = m$.

(b) Determine all positive integers m for which there exists exactly one k with $f(k) = m$.

1994/4: Determine all ordered pairs (m, n) of positive integers such that $\dfrac{n^3+1}{mn-1}$ is an integer.

1994/5: Let S be the set of real numbers strictly greater than -1. Find all functions $f : S \to S$ satisfying the two conditions:

(a) $f(x + f(y) + xf(y)) = y + f(x) + yf(x)$ for all x and y in S;

(b) $\dfrac{f(x)}{x}$ is strictly increasing on each of the intervals $-1 < x < 0$ and $0 < x$.

1994/6: Show that there exists a set A of positive integers with the following property: For any infinite set S of primes there exist two positive integers $m \in A$ and $n \notin A$ each of which is a product of k distinct elements of S for some $k \geq 2$.

1995/1: Let A, B, C, D be four distinct points on a line, in that order. The circles with diameters AC and BD intersect at X and Y. The line XY meets BC at Z. Let P be a point on the line XY other than Z. The line CP intersects the circle with diameter AC at C and M, and the line BP intersects the circle with diameter BD at B and N. Prove that the lines AM, DN, XY are concurrent.

1995/2: Let a, b, c be positive real numbers such that $abc = 1$. Prove that

$$\frac{1}{a^3(b+c)} + \frac{1}{b^3(c+a)} + \frac{1}{c^3(a+b)} \geq \frac{3}{2}.$$

1995/3: Determine all integers $n > 3$ for which there exist n points $A_1, \dots, A_n$ in the plane, no three collinear, and real numbers $r_1, \dots, r_n$ such that for $1 \leq i < j < k \leq n$, the area of $\triangle A_iA_jA_k$ is $r_i + r_j + r_k$.

1995/4: Find the maximum value of x_0 for which there exists a sequence $x_0, x_1, \dots, x_{1995}$ of positive reals with $x_0 = x_{1995}$, such that for $i = 1, \dots, 1995$,

$$x_{i-1} + \frac{2}{x_{i-1}} = 2x_i + \frac{1}{x_i}.$$

1995/5: Let $ABCDEF$ be a convex hexagon with $AB = BC = CD$ and $DE = EF = FA$, such that $\measuredangle BCD = \measuredangle EFA = \pi/3$. Suppose G and H are points in the interior of the hexagon such that $\measuredangle AGB = \measuredangle DHE = 2\pi/3$. Prove that $AG + GB + GH + DH + HE \geq CF$.

1995/6: Let p be an odd prime number. How many p-element subsets A of $\{1, 2, \dots, 2p\}$ are there, the sum of whose elements is divisible by p?

1996/1: We are given a positive integer r and a rectangular board $ABCD$ with dimensions $|AB| = 20, |BC| = 12$. The rectangle is divided into a grid of 20×12 unit squares. The following moves are permitted on the board: one can move from one square to another only if the distance between the centers of the two squares is $\sqrt{r}$. The task is to find a sequence of moves leading from the square with A as a vertex to the square with B as a vertex.

(a) Show that the task cannot be done if r is divisible by 2 or 3.

(b) Prove that the task is possible when $r = 73$.

(c) Can the task be done when $r = 97$?

1996/2: Let P be a point inside triangle ABC such that $\measuredangle APB - \measuredangle ACB = \measuredangle APC - \measuredangle ABC$. Let D, E be the incenters of triangles APB, APC, respectively. Show that AP, BD, CE meet at a point.

1996/3: Let S denote the set of nonnegative integers. Find all functions f from S to itself such that $f(m + f(n)) = f(f(m)) + f(n) \quad \forall\, m, n \in S$.

1996/4: The positive integers a and b are such that the numbers $15a + 16b$ and $16a - 15b$ are both squares of positive integers. What is the least possible value that can be taken on by the smaller of these two squares?

1996/5: Let $ABCDEF$ be a convex hexagon such that AB is parallel to DE, BC is parallel to EF, and CD is parallel to FA. Let R_A, R_C, R_E denote the circumradii of triangles FAB, BCD, DEF, respectively, and let P denote the perimeter of the hexagon. Prove that $R_A + R_C + R_E \geq \dfrac{P}{2}$.

1996/6: Let p, q, n be three positive integers with $p + q < n$. Let $(x_0, x_1, \dots, x_n)$ be an $(n+1)$-tuple of integers satisfying the following conditions:

(a) $x_0 = x_n = 0$.

(b) For each i with $1 \leq i \leq n$, either $x_i - x_{i-1} = p$ or $x_i - x_{i-1} = -q$.

Show that there exist indices $i < j$ with $(i, j) \neq (0, n)$, such that $x_i = x_j$.

1997/1: In the plane the points with integer coordinates are the vertices of unit squares. The squares are colored alternately black and white (as on a chessboard). For any pair of positive integers m and n, consider a right-angled triangle whose vertices have integer coordinates and whose legs, of lengths m and n, lie along edges of the squares.

Let S_1 be the total area of the black part of the triangle and S_2 be the total area of the white part. Let $f(m,n) = |S_1 - S_2|$.

(a) Calculate $f(m,n)$ for all positive integers m and n which are either both even or both odd.

(b) Prove that $f(m,n) \le \frac{1}{2}\max\{m,n\}$ for all m and n.

(c) Show that there is no constant C such that $f(m,n) < C$ for all m and n.

1997/2: The angle at A is the smallest angle of triangle ABC. The points B and C divide the circumcircle of the triangle into two arcs. Let U be an interior point of the arc between B and C which does not contain A. The perpendicular bisectors of AB and AC meet the line AU at V and W, respectively. The lines BV and CW meet at T. Show that $AU = TB + TC$.

1997/3: Let $x_1, x_2, \ldots, x_n$ be real numbers satisfying the conditions

$$|x_1 + x_2 + \cdots + x_n| = 1$$

and $|x_i| \le \dfrac{n+1}{2} \qquad i = 1, 2, \ldots, n$.

Show that there exists a permutation $y_1, y_2, \ldots, y_n$ of $x_1, x_2, \ldots, x_n$ such that

$$|y_1 + 2y_2 + \cdots + ny_n| \le \frac{n+1}{2}.$$

1997/4: An $n \times n$ matrix whose entries come from the set $S = \{1, 2, \ldots, 2n-1\}$ is called a *silver* matrix if, for each $i = 1, 2, \ldots, n$, the ith row and the ith column together contain all elements of S. Show that

(a) there is no silver matrix for $n = 1997$;

(b) silver matrices exist for infinitely many values of n.

1997/5: Find all pairs (a,b) of integers $a, b \ge 1$ that satisfy the equation $a^{b^2} = b^a$.

1997/6: For each positive integer n, let $f(n)$ denote the number of ways of representing n as a sum of powers of 2 with nonnegative integer exponents. Representations which differ only in the ordering of their summands are considered to be the same. For instance, $f(4) = 4$, because the number 4 can be represented in the following four ways $4;\ 2+2;\ 2+1+1;\ 1+1+1+1$.

Prove that, for any integer $n \ge 3$, $2^{n^2/4} < f(2^n) < 2^{n^2/2}$.

1998/1: In the convex quadrilateral $ABCD$, the diagonals AC and BD are perpendicular and the opposite sides AB and DC are not parallel. Suppose that the point P, where the perpendicular bisectors of AB and DC meet, is inside $ABCD$. Prove that $ABCD$ is a cyclic quadrilateral if and only if the triangles ABP and CDP have equal areas.

1998/2: In a competition, there are a contestants and b judges, where $b \ge 3$ is an odd integer. Each judge rates each contestant as either "pass" or "fail". Suppose

k is a number such that, for any two judges, their ratings coincide for at most k contestants. Prove that $k/a \ge (b-1)/(2b)$.

1998/3: For any positive integer n, let $d(n)$ denote the number of positive divisors of n (including 1 and n itself). Determine all positive integers k such that $d(n^2)/d(n) = k$ for some n.

1998/4: Determine all pairs (a,b) of positive integers such that $ab^2 + b + 7$ divides $a^2b + a + b$.

1998/5: Let I be the incenter of triangle ABC. Let the incircle of ABC touch the sides BC, CA, and AB at K, L, and M, respectively. The line through B parallel to MK meets the lines LM and LK at R and S, respectively. Prove that angle RIS is acute.

1998/6: Consider all functions f from the set N of all positive integers into itself satisfying $f(t^2 f(s)) = s(f(t))^2$ for all s and t in N. Determine the least possible value of $f(1998)$.

1999/1: Determine all finite sets S of at least three points in the plane which satisfy the following condition:

- For any two distinct points A and B in S, the perpendicular bisector of the line segment AB is an axis of symmetry for S.

1999/2: Let n be a fixed integer, with $n \ge 2$.

(a) Determine the least constant C such that the inequality

$$\sum_{1 \le i < j \le n} x_i x_j (x_i^2 + x_j^2) \le C \left(\sum_{1 \le i \le n} x_i \right)^4$$

holds for all real numbers $x_1, \ldots, x_n \ge 0$.

(b) For this constant C, determine when equality holds.

1999/3: Consider an $n \times n$ square board, where n is a fixed even positive integer. The board is divided into n^2 unit squares. We say that two different squares on the board are adjacent if they have a common side.

N unit squares on the board are marked in such a way that every square (marked or unmarked) on the board is adjacent to at least one marked square.

Determine the smallest possible value of N.

1999/4: Determine all pairs (n,p) of positive integers such that

- p is a prime,
- n not exceeded $2p$, and
- $(p-1)^n + 1$ is divisible by n^{p-1}.

1999/5: Two circles G_1 and G_2 are contained inside the circle G, and are tangent to G at the distinct points M and N, respectively. G_1 passes through the center of G_2. The line passing through the two points of intersection of G_1 and G_2 meets G at A and B. The lines MA and MB meet G_1 at C and D, respectively. Prove that CD is tangent to G_2.

1999/6: Determine all functions $f : \mathbb{R} \longrightarrow \mathbb{R}$ such that $f(x - f(y)) = f(f(y)) + xf(y) + f(x) - 1$, for all real numbers x, y.

2000/1: AB is tangent to the circles $CAMN$ and $NMBD$. M lies between C and D on the line CD, and CD is parallel to AB. The chords NA and CM meet at P; the chords NB and MD meet at Q. The rays CA and DB meet at E. Prove that $PE = QE$.

2000/2: A, B, C are positive reals with product 1. Prove that

$$(A - 1 + 1/B)(B - 1 + 1/C)(C - 1 + 1/A) \leq 1\,.$$

2000/3: k is a positive real. N is an integer greater than 1. N points are placed on a line, not all coincident. A move is carried out as follows. Pick any two points A and B which are not coincident. Suppose that A lies to the right of B. Replace B by another point B' to the right of A such that $AB' = k\,BA$. For what values of k can we move the points arbitarily far to the right by repeated moves?

2000/4: 100 cards are numbered 1 to 100 (each card different) and placed in 3 boxes (at least one card in each box). How many ways can this be done so that if two boxes are selected and a card is taken from each, then the knowledge of their sum alone is always sufficient to identify the third box?

2000/5: Can we find N divisible by just 2000 different primes, so that N divides $2N + 1$?
[N may be divisible by a prime power.]

2000/6: $A_1A_2A_3$ is an acute-angled triangle. The foot of the altitude from A_i is K_i and the incircle touches the side opposite A_i at L_i. The line K_1K_2 is reflected in the line L_1L_2. Similarly, the line K_2K_3 is reflected in L_2L_3 and K_3K_1 is reflected in L_3L_1. Show that the three new lines form a triangle with vertices on the incircle.

2001/1: ABC is acute-angled. O is its circumcenter. X is the foot of the perpendicular from A to BC. $\measuredangle C \geq \measuredangle B + 30°$. Prove that $\measuredangle A + \measuredangle COX < 90°$.

2001/2: a, b, c are positive reals. Let $a' = \sqrt{a^2 + 8bc}$, $b' = \sqrt{b^2 + 8ca}$, $c' = \sqrt{c^2 + 8ab}$. Prove that $a/a' + b/b' + c/c' \geq 1$.

2001/3: Integers are placed in each of the 441 cells of a 21×21 array. Each row and each column has at most 6 different integers in it. Prove that some integer is in at least 3 rows and at least 3 columns.

2001/4: Let $n_1, n_2, \ldots, n_m$ be integers where m is odd. Let $x = (x_1, \ldots, x_m)$ denote a permutation of the integers $1, 2, \ldots, m$. Let $f(x) = x_1n_1 + x_2n_2 + \cdots + x_mn_m$. Show that for some distinct permutations a, b the difference $f(a) - f(b)$ is a multiple of $m!$.

2001/5: ABC is a triangle. X lies on BC and AX bisects angle A. Y lies on CA and BY bisects angle B. Angle A is $60°$. $AB + BX = AY + YB$. Find all possible values for angle B.

2001/6: $K > L > M > N$ are positive integers such that $KM + LN = (K + L - M + N)(-K + L + M + N)$. Prove that $KL + MN$ is composite.

2002/1: S is the set of all (h, k) with h, k non-negative integers such that $h + k < n$. Each element of S is colored red or blue, so that if (h, k) is red and $h' \leq h, k' \leq k$, then (h', k') is also red. A type 1 subset of S has n blue elements with different first member and a type 2 subset of S has n blue elements with different second member. Show that there are the same number of type 1 and type 2 subsets.

2002/2: BC is a diameter of a circle center O. A is any point on the circle with angle $AOC > 60°$. EF is the chord which is the perpendicular bisector of AO. D is the midpoint of the minor arc AB. The line through O parallel to AD meets AC at J. Show that J is the incenter of $\Delta\,CEF$.

2002/3: Find all pairs of integers $m > 2$, $n > 2$ such that there are infinitely many positive integers k for which $(k^n + k^2 - 1)$ divides $(k^m + k - 1)$.

2002/4: The positive divisors of the integer $n > 1$ are $d_1 < d_2 < \cdots < d_k$, so that $d_1 = 1, d_k = n$. Let $d = d_1d_2 + d_2d_3 + \cdots + d_{k-1}d_k$. Show that $d < n^2$ and find all n for which d divides n^2.

2002/5: Find all real-valued functions f on the reals such that, for all x, y, u, v

$$(f(x) + f(y))(f(u) + f(v)) = f(xu - yv) + f(xv + yu)\,.$$

2002/6: $n > 2$ circles of radius 1 are drawn in the plane so that no line meets more than two of the circles. Their centers are $O_1, O_2, \ldots, O_n$ then

$$\sum_{i<j} \frac{1}{O_iO_j} \leq \frac{(n-1)\pi}{4}\,.$$

2003/1: S is the set $\{1, 2, 3, \ldots, 1000000\}$. Show that for any subset A of S with 101 elements we can find 100 distinct elements x_i of S, such that the sets $\{a + x_i | a \in A\}$ are all pairwise disjoint.

2003/2: Find all pairs (m, n) of positive integers such that $m^2/(2mn^2 - n^3 + 1)$ is a positive integer.

2003/3: A convex hexagon has the property that for any pair of opposite sides the distance between their midpoints is $\sqrt{3}/2$ times the sum of their lengths Show that all the hexagon's angles are equal.

2003/4: $ABCD$ is cyclic. The feet of the perpendicular from D to the lines AB, BC, and CA are P, Q, R, respectively. Show that the angle bisectors of ABC and CDA meet on the line AC iff $RP = RQ$.

2003/5: Given $n > 2$ and reals $x_1 \le x_2 \le \cdots \le x_n$, show that

$$\Big(\sum_{i,j} |x_i - x_j|\Big)^2 \le \frac{2}{3}(n^2-1)\sum_{i,j}(x_i - x_j)^2 .$$

Show that we have equality if and only if the sequence is an arithmetic progression.

2003/6: Show that for each prime p, there exists a prime q such that $n^p - p$ is not divisible by q for any positive integer n.

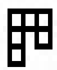
2004/1: Seja ABC um triângulo acutângulo com $AB \ne AC$. A circunferência de diâmetro BC intersecta os lados AB e AC nos pontos M e N, respectivamente. Seja O o ponto médio do lado BC. As bissectrizes dos ângulos BAC e MON intersectam-se em R. Prove que as circunferências circunscritas aos triângulos BMR e CNR têm um ponto em comum que pertence ao lado BC.

2004/2: Determine todos os polinómios P de coeficientes reais que satisfazem a igualdade

$$P(a-b) + P(b-c) + P(c-a) = 2P(a+b+c)$$

para quaisquer números reais a, b, c tais que $ab + bc + ca = 0$.

2004/3: Um gancho é uma figura formada por seis quadrados unitários como no seguinte diagrama



ou qualquer uma das figuras obtidas desta aplicando rotações ou reflexões. Determine todos os rectângulos $m \times n$ que podem ser cobertos com ganchos de modo que:

- o rectângulo é coberto sem buracos e sem sobreposições;
- nenhuma parte de nenhum gancho pode cobrir regiões fora do rectângulo.

2004/4: Seja $n \ge 3$ um inteiro. Sejam $t_1, t_2, \ldots, t_n$ números reais positivos tais que

$$n^2 + 1 > (t_1 + t_2 + \cdots + t_n)(1/t_1 + 1/t_2 + \cdots + 1/t_n) .$$

Mostre que t_i, t_j e t_k são as medidas dos lados de um triângulo para quaisquer i, j, k com $1 \le i < j < k \le n$.

2004/5: Num quadrilátero convexo $ABCD$ a diagonal BD não é bissectriz do Ângulo ABC nem do ângulo CDA. Um ponto P no interior de $ABCD$ satisfaz

$$\angle PBC = \angle DBA \quad \text{e} \quad \angle PDC = \angle BDA .$$

Prove que os vértices do quadrilátero $ABCD$ pertencem a uma mesma circunferência se e só se $AP = CP$.

2004/6: Um inteiro positivo é dito alternante se, na sua representação decimal, quaisquer dois dígitos consecutivos têm paridade diferente.
Determine todos os inteiros positivos n tais que n tem um múltiplo que é alternante.